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ON THE FRACTAL MEASURE AND THE SOJOURN TIME FOR THE PATH OF A LEVY PROCESS IN N-SPACE

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ABSTRACT

After a brief survey of the technique for sojourn time in determining the fractal measure properties of a path of Levy process in n-space, an integral test is established for the maximum difference in order of magnitude between one-side and two-sided lower asymptotic growth for sojourn time process.

1. INTRODUCTION

It is well known, see for example [2,3], that the regularity of Borel subsets of Euclidean space is generally defined by a convergence to 1 of density ratios.

In [16,25] regularity was characterized by the equality between two geometric measures.

For paths of some Levy processes, these sets have such interesting properties of regularity in terms of equality of dimension measures, that are referred to as fractals. Fractal sets arise naturally from the sample paths of stochastic processes and in models appropriate for describing phenomena in a wide range of disciplines. For example, fractals are promising mathematical models for some complicated structures in solid-state physics. This paper considers only those fractal sets, which are trajectories of a Levy process. Our objective here is to describe the use of the local asymptotic growth rates of sojourn time processes as essential tools in analyzing these sets. Furthermore, we attempt to answer the question of the maximum difference in order of magnitude between the lower asymptotic growth rates of the one-side and two-sided sojourn time processes.

In section 2 we introduce the Levy processes in n-space. In section 3 we give a brief discussion on the structure of the path of a Levy process in n-space. Fractal measure for analyzing the path of a Levy processes defined in section 4. Sections 5 and 6 describe the methods of sojourn time processes in calculating the fractal indices and the fractal measures of sample paths, respectively. The main result is in section 7.

2. PRELIMINARIES

A Levy process in n-space is a real-valued function X_t (w) such that each point w of the underlying probability space (Ω,f,P) corresponds to a function $X_t = X_t(w)$ mapping $[0,\infty]$ to R^n, where R^n is the Euclidean space of n dimensions.

As usual in probability theory, we suppress w and denote $X_t(w)$ by X_t. Which is a Markov process with stationary independent increments and characterized by the transition function $f(t, x - x^*)$, relative to the Lebesgue measure in R^n, which is uniquely determined by its Fourier transform

$$\int_{R^n} \exp\{i\langle x, x^*\rangle\} f(t, x^*) dx^* = E \exp\{i\langle x, X_t\rangle\} = \exp - t\,\psi(x)$$

$\psi(x)$ is called the exponent of the process and it has the familiar Levy-Khintchine representation

$$\psi(x) = i\langle x, a\rangle - \frac{1}{2} Q(x) + \int_{R^n} \left[\exp\left(i\langle x, x^*\rangle - 1 - \frac{i\langle x, x^*\rangle}{1 + \|x^*\|^2} \right) \right] V(dx^*)$$

$\langle \cdot,\cdot \rangle$ means the usual inner product in R^n, $\|\ \|$ is the norm in R^n, a is a constant vector in R^n, $Q(.)$, is a non-negative quadratic form and V, a Borel measure in R^n called the Levy measure, satisfies $V(0) = 0$ $V(\infty) < \infty$ and

$$\int_{R^n} \min\left(1, \|x^*\|^2\right) V(dx^*) < \infty$$

We will assume as usual that the Levy processes considered here have been defined so as to have sample paths X_t which are right continuous and have left limits for almost all w in the probability space.

If for every $c > 0$ the process cX_t is another version of X_t, i.e.

$$P\{w : X_t \neq cX_t\} = 0 \text{ for all } t \geq 0$$

we say that process X_t is strictly stable.

For strictly stable processes, Paul Levy [6] determined the form of $\psi(x)$ as

$$\psi(x) = i\langle x, a\rangle - \lambda \|x\|^\alpha \cdot \int_{S^n} J_\infty(x, \theta)\eta(d\theta),\ 0 < \alpha \leq 2, \lambda > 0$$

where

$$J_\alpha(x,\theta) = \begin{cases} \left[1 - i\,\mathrm{sgn}\langle x,\theta\rangle \tan\frac{\pi\alpha}{2}\right]\left\|\left\langle \frac{x}{\|x\|},\theta\right\rangle\right\|^\alpha, & \alpha \neq 1 \\ \left\|\left\langle \frac{x}{\|x\|},\theta\right\rangle\right\| + \left(\frac{2i}{\pi}\right)\langle x,0\rangle \log\|\langle x,0\rangle\| & \alpha = 1 \end{cases}$$

and η is a probability measure on the surface of the unit sphere S_n in R^n. We assume that η is not supported by a proper subspace of R^n and that **α** = 0 so that when η is uniform on S_n, the exponent takes the form $\psi(x) = -\lambda\,\|x\|^\alpha$, $0 < \alpha \le z$, and the resulting process X_t is called symmetric stable process. The symbol α is called the index of the process. The symmetric stable process of index $\alpha = 2$ and $\lambda = \frac{1}{2}$ is the standard Brownian motion, which is characterized by the transition function

$$f(t,x) = (2\pi t)^{-\frac{n}{2}} \exp\left(-\|x\|^2/2t\right)$$

It is only in very few special cases that $f(t,x)$ could be elevated explicitly rather for symmetric stable processes, $f(t,x)$ satisfies the scaling property, i.e. for $r > 0$, $t > 0$, $x \in R^n$,

$$f(t,x) = f\left(rt, r^{1/\alpha}x\right)r^{n/\alpha}$$

so that $r^{-\frac{1}{\alpha}} X_{rt}$ is another version of X_t.

The levy process is an increasing process if each component of X_t is a non-decreasing function for almost all w in (Ω, f, P).

An increasing process on the line is called a subordinator. Increasing processes are characterized by the Laplace transform

$$E\left[\exp -\left\langle X_t, \lambda\right\rangle\right]$$

which, for subordinators, takes the form

$$E\left[\exp -\lambda X_t\right] = \exp\left[-t\varphi(\lambda)\right],\ \lambda \ge 0$$

$$\varphi(\lambda) = c\lambda + \int_{o}^{\infty}\left[1 - \exp(-\lambda x)\right]V(dx)$$

is the exponent of the process. Here $c \geq 0$ and V is the Levy measure satisfying

$$\int_{o}^{\infty} \min(1, x)V(dx) < \infty$$

Levy processes are classified further according to various criteria in [3].
I have not attempted to outline all aspects of levy processes or fractal measures.
I try to describe some concepts on the area of most interest to me.

3. THE STRUCTURE OF THE SAMPLE PATH FOR LEVY PROCESS

For a fixed w in the underlying probability space Ω, we define

$$R_t = \left\{x \in R^n : X_\tau = x \text{ for some } \tau, 0 \leq \tau \leq t\right\}$$

to be the set of points on the sample path in the time interval [0,t], so that when w varies in Ω we obtain a random Borel set $R_t \subset R^n$.

Thus, examining the measure properties of a sample path is equivalent to examining those of the set of points $R_t \subset R^n$ for which a suitable probability measure has been defined.

Similarly, if X_t is a stable process on the line with index $\alpha \in (1,2)$, the corresponding space-time process, called the graph of X_t, is defined by

$$G_s = \left\{\left(X_t, t\right) : 0 \leq t \leq s\right\}$$

which is a random set of points in R^2. see [1,23] for more on these sets.
We shall be interested in almost sure properties of the sample paths, i.e. results about the behaviour of $X_t(w)$ as a function of t which will be valid for almost all sample points w in the underlying probability space Ω.

It is now well known that, for stable processes of index α, the structure of the paths depends on the relationship between the index and the Euclidean dimension see[12], for example. The sample paths for almost all symmetric stable processes such that $n = 1 \leq \alpha \leq 2$ are dense on the line while in the case where $n > \alpha$, almost every path constitutes an uncountable set in R^n having zero Lebesgue measure.

Paul Levy [9] has shown that almost every Brownian path in the plane describes an everywhere dense set, but the two dimensional Lebesgue measure of the set is zero almost surely. Takenchi [19] showed a similar phenomena for Cauchy process on the line where $\alpha = 1 = n$

The Lebesgue measure of the paths in R^n are shown to be zero for n ≥ α

THEOREM [19]

For the symmetric stable paths in R^n

$$P(\|R_\infty\| = 0) = 1 \text{ if } n \ge \alpha$$

$$P(\|R_\infty\| = \infty) = 1 \text{ if } n < \alpha$$

where $\| \ \|$ denotes the n - dimensional Lebesgue measure.

This means that for stable processes of index $\alpha \le n$ Lebesgue measure is no longer the appropriate measure in analyzing these paths.

Thus in order to obtain more precise information about the size of R_t, we need another measure finer than Lebesgue which is called the fractal measure.

4. THE FRACTAL MEASURE:

The fractal measure most studied by probabilists is the Hausdorff measure see [3].

This measure was due to felix Hausdoff [5]. For every Borel subset B of R^n, this measure is defined using economical covers of B by small diameter sets. For the spherical version of this definition, start with the class Φ of functions, $h:(0,1) \to (0,1)$ which are monotone increasing, right continuous with $h(0+) = 0$ and smooth in the sense that $\frac{h(2s)}{h(s)} \le c$ *for finite c and* $0 < s < \frac{1}{2}$.

For every subset $B \subset R^n$, define the set function

$$h - m(B) = \liminf_{\delta \downarrow 0} \left\{ \sum_{i=1}^{\infty} h(2r_i) : B \subset \bigcup_{i=1}^{\infty} B_{ri}(x_i), r_i < \delta \right\}$$

where $B_{ri}(x_i)$ denotes an open ball of radius r_i with center x_i, and the infimum is taken over all possible covers by balls of radius r_i.

This set function is a metric outer measure on the family of all subsets of R^n so that Borel sets will be $h - m()$ measurable.

Clearly $h(s) = s^\beta$ is in Φ for each $\beta > 0$.

Let $h(s) = s^n$, then $h - m()$ on R^n is n-dimensional Lebesgue measure.

Thus to measure the borel subset B of R^n we need $\frac{h_1(s)}{s^n} \to \infty$ as $s \to 0$ so that

if $h_1(s) = s^{\beta}, \beta > 0$.

$h_1 - m(B)$ turns out to be either zero or infinity.

Hence, if B is bounded, there is a number say, $\tilde{\beta} \in [0, n]$, such that

$$s^{\beta} - m(B) = \begin{cases} \infty \text{ for all } \beta < \tilde{\beta}, \beta \geq 0 \\ 0 \text{ for all } \beta > \tilde{\beta}, \beta \geq 0 \end{cases}$$

This means that B has less "substance" than if it were β-dimensional but more "substance" than if it were $\beta - \in$ dimensional for $\in > 0$.

Thus we obtain an index dim B defined by

$$\dim B = \inf\left\{ \beta > 0 : s^{\beta} - m(B) = 0 \right\} = \sup \left\{ \beta > 0 : s^{\beta} - m(B) = \infty \right\}$$

usually called the Hausdoff-Besicovitch dimension of B. "dim" takes the value n on each subset of R^n of positive n-dimensional Lebuegue measure.

If B is countable, the appropriate measure with which to measure it is the counting measure. We consider another fractal measure, whose properties are found in [24], defined by Taylor and Tricot. This measure is called the packing measure.

The packing measure is obtained by seeking to "pack" as many disjoint balls as possible on the set B, where the centers of the balls lie in B.

For $h \in \Phi$, consider

$$h - \tilde{\beta}(B) = \limsup_{\delta \downarrow 0} \left\{ \sum_{i=1}^{\infty} h(2r_i) : Br_i(x_i) \text{ disjoint}, x_i \in B, r_i < \delta \right\}$$

which satisfies all the conditions for a pre-measure but not an outer measure.

A metric outer measure of B is then obtained on R^n by defining

$$h - P(B) = \inf\left\{ \sum h - \tilde{P}\left(B_i\right) : B \subset \cup Bi \right\}$$

$h - P(B)$ is the packing measure of B.

An analogue of the Hausdoff - Besicovitch dimension is the packing dimension of B defined as :

$$Dim\ B = \inf\left\{\beta > 0 : S^{\beta} - P(B) = 0\right\} = sub\left\{\beta > 0 : S^{\beta} - P(B) = \infty\right\}$$

These indices are the fractal indices of the set B and they are distinct. It is well known [see 24] that

$h - m(B) \le h - P(B)$, and since $S^{\beta} - m\left(R^{n}\right) = 0 \; for \; \beta > n$, we can easily see that

$$0 \le \dim B \le Dim B \le n$$

There are alternative valid ways of defining a dimension index see [24, for example]. The notion of Hausdoff-Besicovitch dimension is central to almost all discussions on fractals, though there is no usually accepted definition of the term "fractal".

Taylor [23] refers to those sets B for which dim B = Dim B as fractals. This class of sets includes those sets satisfying B.B Mandebriot's [10,11] definition of fractal sets. We adopt Taylor's definition of fractal sets here. For a systematic study of the main properties of equality of these fractal indices, see [26].

5. THE METHOD OF SOJOURN TIME IN CALCULATING THE FRACTAL INDEX OF A SAMPLE PATH:

For every Borel set B, we use the sample, path X_t to define a random measure

$$m(B) = \left| s \le t : X_s \in B \right\} \Big| \qquad (5.1)$$

called the occupation measure up to time t, which is a projection of Lebesgue measure from [0,t] onto B.

This measure connects the random set R_t formed by the path with the measure of that portion of the time interval [0,t] which the path spends in the set B. It is easy to see that the occupation measure is uniformly spread over R_t.

Let

$$B_r(0) = \left\{ x \in R^n : \|x\| \le r \right\}$$

then

$$T(r) = T(r, w) = \int_{o}^{\infty} I_r(X_t)\,dt$$

$$= m\left(B_r(0)\right)$$

is called the sojourn time process, where I_r, is the indicator function of $B_r(0)$. *If* $\alpha \ge n$, a stable process is recurrent, that means that $\left\{ t : X_t \in B_r(0) \right\}$ is unbounded. In this case T(r) is almost surely infinite. Thus we write $T^S(r)$ for the time spent in the ball $B_r(0)$ by the path X_t up to the time s. If

$x = X_{t_0}, 0 < t_o < 1$, then $m(B_r(x))$, the total time spent by X_t in the ball

$B_r(r)$ up to time 1, is equivalent to the process

$T_1(r) + T_2(r)$ for small r

where T_1, T_2 are independent copies of T.

The first moment of the sojourn time up to time 1. i.e. the expected amount of time the path spends in $B_r(0)$ by the time 1 is

$$ET'(r) = \int_{-\infty}^{\infty} \int_{o}^{1} I_r(X_t) dt \, P(\|X_t\| \leq r)$$

$$= \int_{o}^{1} P(\|X_t\| \leq r) dt \tag{5.2}$$

It is easy to see that (5.2) is close to the reciprocal of the number of balls of radius r needed to cover R_1.

Thus the local asymptotic behaviour of the first moment of the sojourn time process proves very useful in obtaining the fractal indices of the Levy processes.

Pruitt [14] defined, for each Levy process X_t, an index

$$\sigma = \sup\left\{ \beta > 0 : \limsup_{r \downarrow 0} \frac{ET'(r)}{r^\beta} < \infty \right\}$$

and shown that dim $R_1 = \sigma$ a.s. while Hendricks in [6] considered the index

$$\tilde{\sigma} = \sup\left\{ \beta > 0 : \liminf_{r \downarrow 0} \frac{ET'(r)}{r^\beta} = 0 \right\}$$

and in [22], Taylor proved that $Dim\ R_1 = \tilde{\sigma}$ a.s. for a Levy process in-space. The fractal indices are now known for general processes with respect to their sojourn time processes.

For a stable process X_t of the index α,

$$\dim R_1 = Dim\ R_1 = \min(\alpha, n)\ a.s.$$

so that, in Taylor's sense, R_1 is a random fractal arising naturally from the path of a stable process in n -space.

6. SOJOURN TIME AND THE FRACTAL MEASURE OF SAMPLE PATHS

Typical paths of Levy processes have zero fractal measures in their dimensional spaces.

To obtain further information about the "size" of the path, we consider the "correct" measure function $h \in \Phi$ for which the fractal measure is both positive and finite.

For random sets, the correct $h \in \Phi$ is often complicated when it exists. The use of local asymptotic behaviour of the occupation measure m has been found to be very helpful in obtaining the "correct" $h \in \Phi$.

Consider The Limit Of The Density Ratios:

Let m be a Borel measure on R^n *and* $h \in \Phi$. Define the spherical densities of $\tilde{m}$ *at x by*

$$D_h \tilde{m}(x) = \limsup_{r \downarrow 0} \frac{\tilde{m}\left(B_r(x)\right)}{h(2r)} \tag{6.1}$$

and

$$\hat{D}_h \tilde{m}(x) = \liminf_{r \downarrow 0} \frac{\tilde{m}\left(B_r(x)\right)}{h(2r)} \tag{6.2}$$

(6.1) and (6.2) are the upper and lower spherical densities of $\tilde{m}$ respectively. Is it easy to see that the upper spherical density of the measure m at 0 has a simple meaning in terms of the sojourn time for it becomes

$$\limsup_{r \downarrow 0} \frac{T(r)}{h(2r)} \tag{6.3}$$

while the lower spherical density of m at X_{t_0} ,0< to< 1, becomes

$$\liminf_{r \downarrow 0} \frac{T_1(r) + T_2(r)}{h(2r)} \qquad (6.4)$$

The density theorems of Rogers and Taylor [18] have been useful in analyzing the measure m using the local densities.

Applying the density theorems:

THEOREM 2

There exist constants z_1, z_2 depending on *h and n*, such that for any

Borel set $B \subset R^n$ containing x, and any finite Borel measure $\tilde{m}$, we have

$$z_1 \tilde{m}(B) \left\{ D_h \tilde{m}(x) \right\} \leq h - m(B)$$

$$\leq z_2 \tilde{m}\left(R^n\right) \sup_{x \in B} \left\{ D_h \tilde{m}(x) \right\}$$

and

$$Z_1 \tilde{m}(B) \inf_{x \in B} \left\{ \liminf_{r \downarrow 0} \frac{h(2r)}{\tilde{m}\left(B_r(x)\right)} \right\} \leq h - P(B)$$

$$\leq \tilde{m}\left(R^n\right) \sup_{x \in B} \left\{ \limsup_{r \downarrow 0} \frac{h(2r)}{\tilde{m}\left(B_r(x)\right)} \right\}$$

we have that the correct function h ∈ Φ such that.

$$\limsup_{r \downarrow 0} \frac{T(r)}{h(2r)} = C_1$$

gives

$$h - m\left(R_t\right) \geq C_2 > 0$$

while the function h∈φ for which

$$\liminf_{r \downarrow 0} \frac{T_1(r) + T_2(r)}{h(2r)} = C_3$$

gives $h - P(R_t) \geq c_4 > 0$.

The exceptional time sets involved in the computation of the limits (6.3) and (6.4), though have zero Lebesgue measures, may well contribute to the fractal measures. Hence to complete the proofs for the upper bounds for these fractal measures, there is still detailed work to be done so as show that the fractal measures of these "bad" points (exceptional time sets) are zeros. See [1].

The technique outlined above always give the correct $h \in \Phi$ for measuring the sample paths.

The sojourn time for the graph G_S in the rectangle $\{\|X_t\| \leq r, 0 \leq t \leq r\}$ is given by

$$T^r(r) = \int_o^a I_{(-r,r)} \quad XS\, ds$$

Once we know that $h - m(R_t)$ *and* $h - P(R_t)$ are bounded, being continuous subordinators, there exist finite constants c_6 and c_8 such that $h - m(R_t) = c_6 t$ *and* $h - P(R_t) = c_7 t$ respectively similar method works for the graph of the stable processes see[17].

These function $h \in \Phi$ are known for almost all paths of Levy processes see [23].

THE MAIN RESULT:

In a paper [13] with O.O. Ugbebor, we investigated the lower asymptotic behaviour of $T(r)$ for symmetric stable processes in n-space of index $\alpha > 1$. It is not surprising that the lower asymptotic growth rate of $T_1(r) + T_2(r)$ is of larger "order" than that of $T(r)$ for small values of r. This brings us to the question: what is their minimum difference in order of magnitude?

This problem was suggested to me (in private communication) by professor S.J.Taylor. We show something more general, namely Theorem 3.

Theorem 3

For an increasing process y_t with continuous paths, if

$$\int_{or} \frac{dt}{t\varphi(t)} < \infty \; and \; \liminf_{t \downarrow 0} \frac{y_t}{h(t)} < \infty$$

then

$$\liminf_{t \downarrow 0} \frac{y_t^1 + y_t^2}{h(t)\varphi(t)} = 0$$

where y_t^1, y_t^2 are independent copies of y_t, and h, φ are monotone **functions of** t.

PROOF:

Suppose

$$\int_{0+} \frac{dt}{t\varphi(t)} < \infty$$

For any fixed finite $\lambda, 0 < \delta < 1 \; and \; \hat{\delta} = \max(1, 1 - \delta) \; set \; a_k = \rho^{-k}, \rho > 1$ and define

$$D_k = \left\{ y_{a_{k+1}} > \hat{\delta}\lambda h(a_k) \right\}$$

and

$$E_k = \left\{ y^1_{a_{k+1}} + y^2_{a_{k+1}} > \lambda h(a_k) \right\}$$

$$\subset \left\{ y^1_{a_{k+1}} > \delta\lambda h(a_k) \right\} \cup \left\{ y^2_{a_{k+1}} > (1 - \delta)\lambda h(a_k) \right\}$$

so that $P(E_k) \leq c\, P(D_k)$, c is a constant

If $\sum P\left(D_k\right) < \infty$, *then* $P\left(D_k\right) = 0$ by Borel-Cantelli lemma, so that $P\left(E_k\right) = 0$. Thus a.s. D_k for at most a finite number of k for each λ and hence we find $t \in \left[a_{k+1}, a_k\right]$ for which

$$\liminf_{t\downarrow} \frac{y_t}{h(t)} < \infty$$

And if

$$\sum \frac{1}{\varphi\left(a_k\right)} = \int_{0+} \frac{dt}{t\varphi(t)} < \infty$$

then

$$\liminf_{k\to\infty} \frac{1}{\varphi\left(a_k\right)} = 0$$

so that

$$\liminf_{t\downarrow 0} \frac{y_t^1 + y_t^2}{h(t)\varphi(t)} = 0$$

Our application is to the time spent in a ball of radius t.

REFERENCES:

[1] Adler, R.J. "The geometry of random fields" Wiley, 1980.

[2] Besicovith, A.S, "On the fundamental properties of linearly measurable plane sets of points". Math. Ann. 98(1929),422-464.

[3] Falconer, K.J. "The geometry of fractal sets". Cambridge University Press (1985).

[4] Fristedt, B.E. "Sample functions of stochastic processes with stationary independent

increments". Advances in probability, vol.3, Marcel Dekker, New York (1973).

[5] Hausdoff, F. "Dimensio and aussers Mass". Math. Ann. 79 (1919), 157-179.

[6] Hendricks, W.J. "A uniform lower bound for Hausdoff dimension for transient symmetric Levy Processes" Annals of Prob. 11(1984), 589-592.

[7] Levy, P. "Theorie de Paddition des variables aleatoires". Paris, 1937.

[8] "Le mouvement Brownian plan". Amer. J.Math. 62(1940),487-550

[9] "Processus sto chastiques et mouvements Brownian". Pairs, (1948)

[10] Mandelbrot, B.B. "Fractals: form, chance and dimension". Freeman (1977).

[11] "The fractal geometry of nature". Freeman (1982).

[12] Mekean, H.P. "Sample functions of stable processes". Ann. of Math. 61(1955,56-579.

[13] Okoroafor, A.C. and Ugbebor, O.O. "Lower asymptotic behaviour of the sojourn time for a stable process" in contemporary stochastic analysis, proceedings Ibadan (1989) ed. Ekhaguere World Scientific Press (1991), 109-126.

[14] Pruit, W.E. "The Hausdoff dimension of the range of a processes with stationary independent increments". Journ. Math. and Mech. 19 (1969). 371-378.

[15] Pruit, W.E. and Taylor, S.J. "Sample path properties of processes with stable components". Wahr. Verw-Geb 12, (1969), 267-289.

[16] Raymon, X. and Tricot, C. "Packing regularity of sets in n-space". Math Proc. Camb. Phli. Soc. (1988), 103-145.

[17] Rezathanlou, F. and Taylor, S.J. "The packing measure of the graph of stable process", Prepriut, 1987.

[18] Rogers, C.A. and Taylor, S.J. "Functions continuous and singular with respect to a Hausdorff measure". Mathematika 8(1961), 1-31.

[19] Takeuchi, S.J. On the sample paths of symmetric stable processes in space", J. Math. Soc. Japan Vol. 16, No.2 (1964), 109-127.

[20] Taylor, S.J. "Sample path properties of processes of atransient stable process". J. Math and Mechanic, 16(1967), 1229-1246.

[21] ______ "Sample path properties of processes with stationary independent increments". Stochastic geometry and analysis, London (1972).

[22]_____ "The use of packing measure in the analysis of random sets". Proc. of 15th Symposium on stochastic processes (1985), 214-222.

[23] ____ "The measure theory of random fractal". Math. Proc. Camb. Phli. Soc. 100(1986), 383-406.

[24] ____ and Tricot, C. "Packing measure and its evaluation for Brownian path". Trans. Amer. Maths. Soc. 288 (1985), 679-699.

[25] ____ "Packing measure of rectifiable subsets of the plane". Math. Proc. Camb. Phli. Soc. 99 (1986), 285-296.

[26] Tricot, C. "Two definitions of fractional dimension". Math. Proc. Camb. Phli. Soc. (1982), 91, 57-74.