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A LEAST SQUARES FIT OF MESON SPECTRA IN THE DIRAC EQUATION WITH A POWER-LAW POTENTIAL MODEL

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ABSTRACT

The recent discovery of the top quark and the huge amount of data generated among various particle physics data groups has generated a lot of interestamong researchers. This calls for an update of our potential models describing these systems.using a single power – law potential of the form:

$V(r) = Ar^{1/\alpha} - V_0$

where A, V_0 and ($\alpha > 0$) are parameters to be fitted, we have performed a least squares fit by fitting all the spectra of both light and heavy mesons (including leptonic decay-widths) in the Dirac equation. We have found for each meson type the optimum values of the parameters of our potential model described above that are required to give a good description of the meson spectra through a least squares fit.

We next used the same potential model to predict the mass-spectra of the toponium (tt) system. The experimental data for this system is still in its infancy but we hope our potential model developed here will help in the further understanding of this meson structure.

1. INTRODUCTION:

The discovery of the top quark and the tremendous development in the particle physics accelerator technology (Campagnari et al., (1997)) has generated an immense interest among researchers. In effect a huge amount of data has been obtained by various particle physics data groups (Abe et al., (1999)) and this has called for an update of our models describing these systems. This updating in our models is particularly necessary because even though quantum chromodynamics (QCD) theory has long since been identified as the correct underlying theory of strong interactions at the basic structural level of hadrons, it cannot explain many low-lying phenomena such as M1 transitions. Thus other attempts have to be made to construct phenomenological models, incorporating the basic features of QCD. The relativistic quark model (Godfrey et al., (1985)) provides a phenomenological frame-work to describe all mesons in a unified way. The application of the bag-model to radiative decays proposed by Hays et al., (1978) gives rather a poor agreement with the experimental data The failure of the MIT-bag-model can be traced back to be basically due to underestimation of the static magnetic moments as well as the transition moments of the hadrons. In the so-called calculation of the transition moment, the region of the overlap integral is ambiguous because of the different radii of the initial and the final state bags. The cloudy bag model (CBM) – a hybrid version of the bag model endowed with chiral symmetry proves to be a better scheme to deal with this kind of problem. The analysis of Singer et al., (1986) based on CBM approach provides a completely new picture of radiative decay rates to be relatively in better agreement with observed values.

The other attempt used in describing meson properties has been to use potential models (Jena et al., (1999)). Using such a model we have recently been able to describe all meson spectra including leptonic decay-widths of all light and heavy mesons (Sharma and Fiase (2000)) using a power-law potential model. The aim of this paper is to extend the work of Sharma and Fiase (2000) by performing a least squares fit to all the spectra of both light and heavy mesons (including leptonic decay - widths) in the Dirac equation using a power-law potential. This work is different from the previous work of Sharma and Fiase (2000) because in the previous work, only one single set of data points (the experimental ground state energy in this case) was used to fit the potential parameters described above, whereas in the present work we require that all the experimental energy levels (including the experimental ground state energy) be used in fitting the potential parameters described above through a least squares fit. With this procedure we hope to obtain an improvement over the previous work of Sharma and Fiase (2000) since more data points are used in fitting the parameters of our potential model, thus hopefully providing a more accurate potential model for the meson system. As will be shown later, we have found for each meson type, the optimum values of the parameters of our potential model described above which enables us to give a good description of the meson spectra through a least squares fit.

We consider this approach to be quite useful because in the absence of a fundamental theory to explain subatomic phenomenon appropriately, this is the only way that huge amount of data generated among particle physics data groups could be very accurately described. We have next used this approach to determine the mass-spectra for the $(t\bar{t})$ bound states. Even though the experimental data for this system is still in its infancy, we hope our potential model developed here will help in the further understanding of this meson structure.

2. THE POTENTIAL MODEL AND THE DIRAC BOUND-STATES.

In order to describe meson properties we have chosen a potential of the form:

$$V(r) = Ar^{1/\alpha} - V_0 \qquad (1)$$

where A, V_0 and α are parameters to be fitted. We next write the Dirac equation in independent particle model of quarks.

Following the procedure suggested by Magyari (1980), V(r) may be rewritten as:

$$V'(x) = \frac{1}{2}V(r) = V_s(r) + V_v(r), \quad (2)$$

with the choice of the vector fraction $g_v = \frac{1}{2}$.

In this case each of the scalar and vector parts would equal $\frac{1}{4}V(r)$. Now Dirac equation may be written as $(\hbar = c = 1)$:

$$\left(\vec{\alpha}\cdot\vec{p} + m_q\beta\right)\psi(r) = \left[E' - V_v(r) - V_s(r)\beta\right]\psi(r). \quad (3)$$

where $V_v(r)$ and $V_s(r)$ are spherically symmetric. Following Schiff (1968), we may separate equation (3) into a system of the following coupled equations for radial functions, $\phi(r)$ and $\chi(r)$: .

$$\begin{aligned} &\left(E' - V_v - V_s - m_q\right)\phi(r) + \left(\frac{k+1}{r} + \frac{d}{dr}\right)\chi(r) = 0 \\ &\left(E' - V_v + V_s + m_q\right)\chi(r) + \left(\frac{k-1}{r} - \frac{d}{dr}\right)\phi(r) = 0, \end{aligned} \quad (4)$$

where k = L + 1, if the total angular momentum of the quark is j = L+ ½ and k = -1, if j = L - ½.

Substituting $\phi(r) = \frac{U(r)}{r}$ and $V_s(r) = V_v(r) = \frac{1}{4}\left\{Ar^{1/\alpha} - V_o\right\}$, the equation for large component $\varphi(r)$ takes the form (Magyari (1980)):

$$\frac{d^2}{dr^2}U(r) + \left[\left(E' + m_q\right)\left(E' - m_q - 2V_s(r)\right) - \frac{l(l+1)}{r^2}\right]U(r) = 0 \quad (5)$$

If we choose $\rho = \frac{r}{r_0}$ and $r_o = \left[\frac{\left(m_q + E'\right)A}{2}\right]^{\frac{-1}{2+(1/\alpha)}}$

equation (5) reduces to the Schroedinger – like equation with a power-law potential of the form:

$$\frac{d^2 U(\rho)}{d^2 \rho} + \left[\varepsilon - \rho^{\frac{1}{\alpha}} - \frac{l(l+1)}{\rho^2} \right] U(\rho) = 0 \qquad (6)$$

with

$$\varepsilon = \left(E' - m_q + \frac{1}{2} V_o \right) \left[\left(m_q + E' \right) \left(\frac{2}{A} \right)^{2\alpha} \right]^{\frac{1}{1+2\alpha}} \qquad (7)$$

Equation (6) is a Schroedinger – like equation which can be solved using semi-classical methods of Quigg and Rosner (1977). Our procedure is similar to the one used by Barik and Barik (1981). Therefore, equation for the Dirac bound state masses can be written as (Barik and Barik(1981)):

$$M_{nl}(q\bar{q}) = 2ax_{nl} + 2m_q - V_o, \qquad (8)$$

$a = (A)^{1/(1+1/\alpha)}$ and χ_{nl} is the positive root of the following equation:

$$x_{nl}^{[1+2\alpha]} \left(x_{nl} + b \right) = 2^{-2\alpha} \left(\varepsilon_{nl} \right)^{1+2\alpha} \qquad (9)$$

where

$$\varepsilon_{nl} = \left[\left(n + \frac{l}{2} - \frac{1}{4} \right) \frac{2\sqrt{\pi}\,\Gamma(3/2+\alpha)}{\Gamma(1+\alpha)} \right]^{\frac{2}{1+2\alpha}} \qquad (10)$$

and

$$b = \left(\frac{2m_q - \frac{1}{2} V_o}{a} \right).$$

3 RATIO OF LEPTONIC DECAY – WIDTHS

The stringent test that must be passed by any potential model if it is to be a good model is that it should be able to reproduce electromagnetic properties such as leptonic decay-widths satisfactorily. We define the leptonic-width ratio as (Jena (1983a):

$$R_{ns} = \left\{ \left[\frac{M_{ls}\left(q\bar{q}\right)}{M_{ns}\left(q\bar{q}\right)} \right]^2 \right\} \left| \frac{\psi_{ns}(0)}{\psi_{ls}(0)} \right|^2 \tag{11}$$

where (Barik and Barik (1981):

$$\left|\psi_{ns}(0)\right|^2 = \frac{\left(m_q\right)^{\frac{3}{2}} \left(E_{ns}\right)^{\frac{1}{2}} \left(\frac{dE_{ns}}{dn}\right)}{4\pi^2} \tag{12}$$

In equations (11) and (12), $\psi_{ks}(0)$ is the bound – state wave function of the kth state of the quark – antiquark system evaluated at the origin, $M_{ks}\left(q\bar{q}\right)$ is the mass of the meson under consideration and E_{ns} is the excitation energy. From equation (8), (9) and (10) we obtain:

$$\left|\psi_{ns}(0)\right|^2 = \frac{\dfrac{(m_q)^{3/2}(a)^{3/2}\, 2^{(5/2-2\alpha)}\left(n-1/4\right)G(\alpha)}{4\pi^2}}{X_{ns}^{(2\alpha-1/2)}\left\{2(1+\alpha)X_{ns}+(1+2\alpha)b\right\}} \tag{13}$$

where

$$G(\alpha) = \frac{2(\pi)^{1/2}\,\Gamma(3/2+\alpha)}{\Gamma(1+\alpha)}. \tag{14}$$

Substituting equation (13) in equation (11), the expression for R_{ns} gives:

$$R_{ns} = \frac{\left(\dfrac{M_{ls}(q\bar{q})}{M_{ns}(q\bar{q})}\right)^2 (4n-1)\, X_{ls}^{(2\alpha-1/2)}\left\{2(1+\alpha)X_{ns}+(1+2\alpha)b\right\}}{\left\{2(1+\alpha)X_{ns}+(1+2\alpha)b\right\} 3X_{ns}^{(2\alpha-1/2)}}. \tag{15}$$

4 THE LEAST SQUARE FITTING PROCEDURE.

In this section we describe the least squares fitting procedure used in obtaining the best – fit parameters for our chosen potential. For each meson type, we define the variance, χ_F which gives a quantitative estimate of the quality of fit as follows:

$$\chi_F = \sum_i \frac{\left(q_{cal}(i) - q_{\exp}(i)\right)^2}{\left(q_{\exp}(i)\right)^2} \tag{16}$$

where the $q_{cal}(i)$ denote the calculated energy levels including the letonic decay-widths for each of the meson system while the $q_{\exp}(i)$ are their experimental counterparts. The parameters of our potential model are then chosen such that the variance χ_F takes on a minimum value for each of the meson systems. These optimized set of parameters give the best agreement of our calculated energy levels and leptonic decay-widths with experimental data.

5. THE RESULTS:

We now present the results of our findings based on the following procedure. Using the information on observed and calculated masses including the leptonic decay-widths we have fitted the potentials $V_c(r)$ for the charmonium $(c\bar{c})$, $V_B(r)$ for the bottomonium $(b\bar{b})$ *and* $V_s(r)$ for the strange quark-anti quark $(s\bar{s})$ meson systems through a least square fit which are approximately given by:

$$V_c(r) = 2.65r^{1/4.0} - 4.2 \tag{17}$$

$$V_B(r) = 6.72r^{1/10.0} - 7.54 \tag{18}$$

and

$$V_s(r) = 0.29r - 1.25, \tag{19}$$

Where B,C and S correspond to the bottomonium, the charmonium and the strange quark-anti quark meson systems respectively. For each value of α, we have plotted the variance χ_F verses Vo and have searched for the minimum value of the variance which will then give us the optimum set of the parameters A and Vo. As can be seen from Figures 1-3 the variances for each meson system varies smoothly with Vo decreasing smoothly until a minimum value of the variance is reached then it increases again.

In Table I, we present the fitted energy levels of the charmonium $(c\bar{c})$, the bottomonium $(b\bar{b})$ and the strange quark – anti quark $(s\bar{s})$ meson systems based on equations (16) to (19) above and compare them with the experimental values and the results of other workers. From our results, we note that the agreement with the experimental data and the values quoted by other authors is quite reasonable.

In Table II, we present the results of our fitted leptonic decay-widths for the charmonium $(c\bar{c})$, the bottomonium $(b\bar{b})$ and the strange quark-anti quark $(s\bar{s})$ meson systems and compare them with the corresponding experimental ratios and the results of other workers on the same meson systems. We have found that our fitted data are in excellent agreement with experiment and work of previous authors on these systems.

It is interesting to note that the optimum values of α obtained for each of the meson system is approximately equal to twice the mass of the meson system under consideration. For example, the experimental masses for the $(s\bar{s})$, $(c\bar{c})$ and the $(b\bar{b})$ meson systems are approximately, 0.5, 2.0 and 5 GeV respectively corresponding to the values of α =1.0, 4.0 and 10.0 respectively which are exactly the values of α found in equations (17) – (19). Furthermore, one should note that the heavier the meson system the larger the value of A and the corresponding Vo. We found these observations in our earlier work (Sharma and Fiase (2000)) even though we had used only one experimental data point to fit the mass spectra of each of the meson systems which should however not be considered as accurate as in the present case where we used all the experimental data points in our fitting programme.

Based on these observations, we should expect larger values of α , A and Vo for the very heavy meson systems. Recently the top quark with the experimental mass of m_t =176 GeV (Abe et al. (1999)) has been discovered. Evidence of the existence of its bound quark-anti quark $(t\bar{t})$ system has also recently been reported (Abbott et al. (1999)), but so far, its experimental mass spectra data is not yet available with which we could compare our results. However, borrowing the idea from our results for both the lighter $(s\bar{s})$ and the heavier $(b\bar{b})$ systems, we expect the values of Vo and A to be very large for this very heavy meson system. For example, we expect the values of Vo and A to be greater than the heaviest of the meson system so far calculated which is the $(b\bar{b})$ system in this case having Vo, A=7.54, 6.72. In that case we have calculated several ranges of energy spectra for the $(t\bar{t})$ system with variation in the values of the parameters Vo, and A presented in Table III. We have set the lowest limit of our parameters to be Vo, A=7.0, 6.99 corresponding approximately to those of the $(b\bar{b})$ meson system. Whereas, our highest limit corresponds to the maximum values of Vo and A our computer programme can generate which are Vo, A=160, 160.55 in the present case. Here we take $a = 2m_t$ based on our observations above. Interestingly, we have found that the spacing of the $(t\bar{t})$ system is rather insensitive to the variation in the parameters and the overall spectra of the $(t\bar{t})$ system is rather compressed. Thus, it could be conjectured from our modest analyses that the $(t\bar{t})$ system mass spectra when eventually determined experimentally, will be rather compressed much more compared to that of the $(c\bar{c})$ or $(b\bar{b})$ bound states which both have the 2S and 1S level spacing as high as 0.6GeV compared to 0.017 – 0.38 GeV found in the present analysis.

6 CONCLUSION

We have fitted the mass-spectra and ratios of the decay-widths, using a power-law potential in the Dirac equation. Our fitted data for both the $(s\bar{s})$, the $(c\bar{c})$ and the $(b\bar{b})$ systems are in good agreement with the experimental data as well as with the calculations of other workers. Thus, in spite of its discomforting non-coulombic nature which contradicts the predictions of the QCD, our simple power-law potential is capable of fitting the spectra of the $(s\bar{s})$, the $(c\bar{c})$ and the $(b\bar{b})$ system simultaneously.

Encouraged by our approach, we have also predicted the mass-spectra for the $(t\bar{t})$ system with variation in the parameters used in our defining equations. Since there is yet no experimental data available for the $(t\bar{t})$ system with which to compare our results, we have thus used a wide range of these parameters to study the sensitivity of the energy level spacing for the given system. Our results reveal an interesting feature that over a wide range of chosen parameters, the energy level spacing for the $(t\bar{t})$ system is rather compressed compared to $(c\bar{c})$ and $(b\bar{b})$ system. It will now be interesting to watch the experimental evolution of this system, both as a test of our ideas of meson structure and as a guide towards the understanding of the rather elusive nature of such phenomenon as the quark confinement in hadronic system.

Table I. Comparison of fitted energy levels with other potential models and experimental data for the $(s\bar{s})$, $(c\bar{c})$ and $(b\bar{b})$ systems.

$(c\bar{c})$ states

state	Data[Caso et al.(1998)] (MeV)	Calculated	[Zhang et al.(1991)]	[Jena (1983a)]
1S	3096.88 ± 0.04	3097.0	3114	3097
2S	3686.00 ± 0.09	3705.84	3690	3686
3S	4040 ± 10	4075.37	4040	4031
4S	4415 ± 6	4348.96	4320	4278

$(b\bar{b})$ states

state	Data[Caso et al. (1998)]	Calculated	[Zhag et al.(1991)]	[Grant et al. (1993)]
1S	9460.37 ± 0.21	9460.0	9460	9423
2S	10023. 30 ± 0.31	10020.1	10000	100423
3S	10355.3 ± 0.5	10337.0	10320	10358
4S	10580 ± 3.5	10562.1	10570	10567

$(s\bar{s})$ states

state	Data[Caso et al.(1998)]	Calculated	[Gara et al. (1990)]	[Jena (1983a)]
1S	1019.413 ± 0.008	1020	1098	1019.6
2S	1680 ± 0.20	1689.9	1616	1640.0
3S	-	2181.9	-	2004.0
4S	-	2589.8	-	2264.0

Table II. Ratios of fitted leptonic decay-widths compared with the experimental data for $(b\bar{b})$, $(c\bar{c})$ and $(s\bar{s})$ systems.

	$(s\bar{s})$		$(c\bar{c})$		$(b\bar{b})$	
state	Calc.	Jena (1983a)	Calc.	Data[Caso et al., (1998)]	Calc.	Data[Casoet al.(1998)]
1S	1.00	1.00	1.00	1.00	1.00	1.00
2S	0.325	0.194	0.424	0.45±0.09	0.453	0.44±.06
3S	0.116	0.089	0.171	0.16±0.10	0.189	0.35±0.04
4S	0.057	0.054	0.091	0.09±0.06	0.104	0.20±0.06

Table III. Predicted mass – spectra for the $(t\bar{t})$ system. Here α = 2m_t has been taken to equal roughly the $(t\bar{t})$ systems in the 1S state. We take m_t=176 GeV.

State	V_0,A (7.0,6.99)	V_0,A (15.0,15.00)	V_0,A (25.0,25.02)	V_0,A (120.0,120.36)	V_0,A (160.0,160.55)
1S	352.00	352.00	352.000	352.00	352.00
2S	352,017	352.036	352.060	352.29	352.38
3S	352.025	352.055	352.092	352.44	352.59
4S	352.032	352.069	352.114	352.55	352.73

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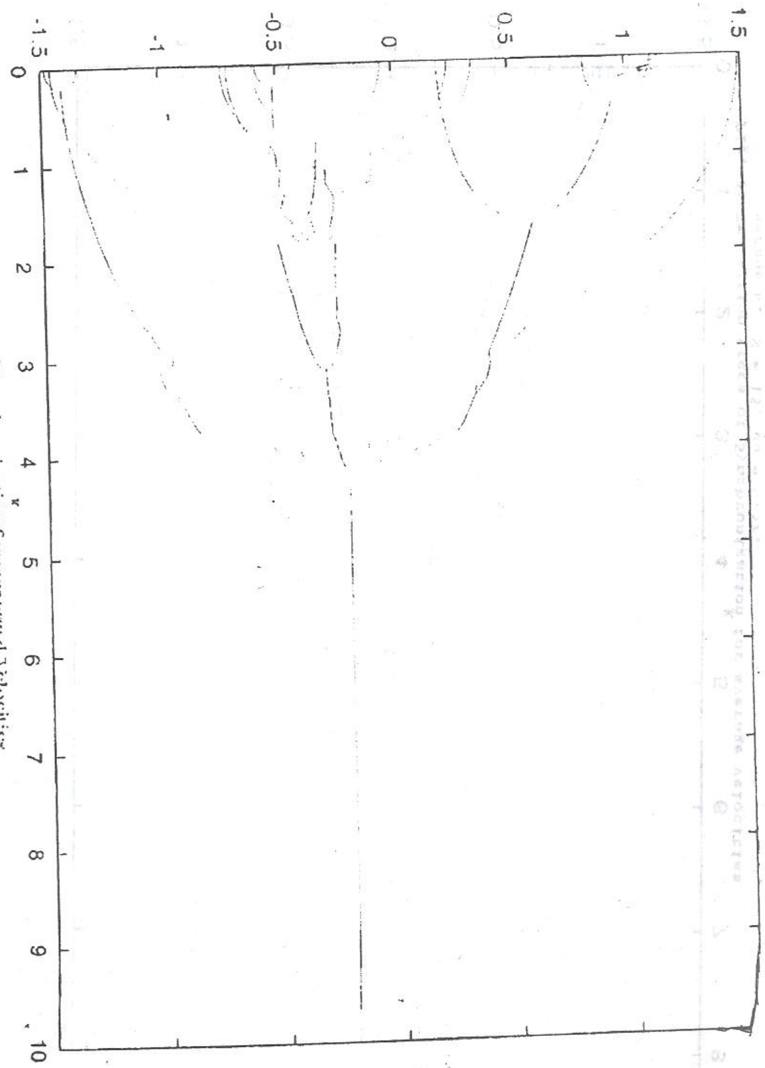

Fig. 1: Transition trees of Synchronization for averaged Velocities versus K,N = 15, W0 = 1.77.

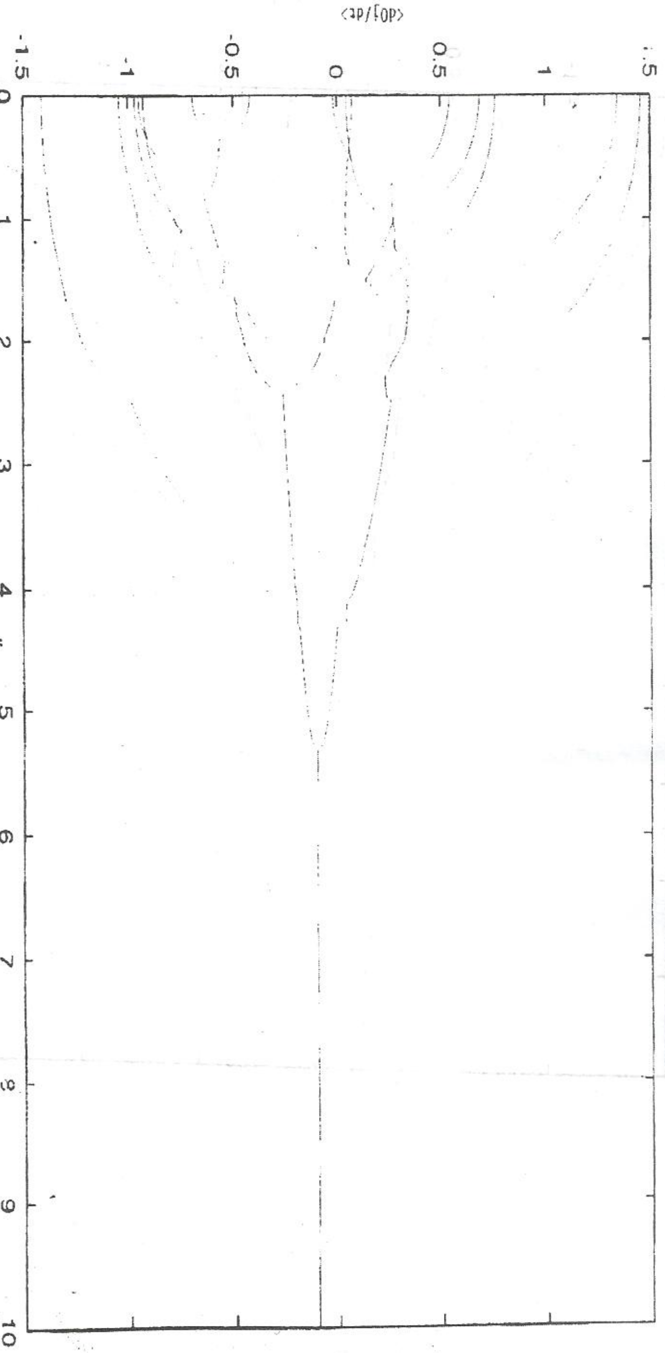

Fig. 2: Transition trees of Synchronization for average velocities versus K. N = 15, ωo = 0.52.