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EXACT EIGENENERGIES AND EIGENFUNCTIONS OF THE KP^{2M} ANHARMONIC OSCILLATOR

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ABSTRACT

In this paper we derive the exact analytic eigenenergies and eigenfunctions for the relativistic linear harmonic oscillator for comparison with the corresponding results obtained earlier by perturbation theoretic schemes.

1. INTRODUCTION

The great interest in the generalizations of the Schrodinger's quantum mechanical wave equation for the linear harmonic oscillator is very well known. It is well known how to analyse the linear simple harmonic oscillator (ISHO) for the cubic and quartic position dependent perturbation potentials with the classic perturbation theory [1,2]. In the year 1961 Epstein and Hirschfelder[3] applied the hypervirial and Feyman theorems in perturbation to the anharmonic oscillator with a perturbation of the general form kx^m, where k is an infinitesimal parameter and x is the position coordinate. Then Maduemezia[4] extended the hypervirial and Feynman approximation theorems to the harmonic oscillator with velocity – dependent perturbation of the general form kp^{2m}, where p is the momentum coordinate variable. And as an example[5] he used that theory to estimate the eigenenergies of the relativistic harmonic oscillator defined by the hamiltonian

$$H = \left(p^2c^2 + m_0^2c^4\right)^{\frac{1}{2}} - m_0c^2 + \frac{1}{2}m_0 w_0{}^2 x^2$$

or

$$H \approx \frac{1}{2}m_o w_o^2 x^2 + \frac{p^2}{2m_o} - \frac{p^4}{8m_0 c^2} \qquad (1)$$

As a follow-up of these developments we derive in this paper the exact analytic eigenfunctions of the relativistic linear harmonic oscillator with hamiltonian (1) for comparison with the approximate solutions obtained earlier. Towards this goal we note that the quantum mechanical eigenproblem for the relativistic oscillator (2) with hamiltonian of fourth order in , is given[4,5] by

$$0 = \left\{ \begin{array}{l} \frac{\hbar^4}{8m_o^3c^2} U_4''''(x) + \frac{\hbar^2}{2m_o} U_4''(x) \\ + \left[E_4 - \frac{1}{2} m_o \omega_o^2 x^2 \right] U_4(x) \end{array} \right\} \tag{2}$$

where E_4 and U_4 are the quantum mechanical energy and corresponding wave function respectively, correct to the order of $\hbar^{-4}$, all other symbols have their usual meanings. This equation may be written equivalently and more conveniently as

$$0 = \left\{ \begin{array}{l} \in_4 U_4''''(x) + \in_2 U_4''(x) \\ + \left[(E_4 - m_o c^2) - \frac{1}{2} m_o \omega_o^2 x^2 \right] U_4(x) \end{array} \right\} \tag{3}$$

where

$$\in_2 = \frac{\hbar^2}{2m_o} \tag{4}$$

and

$$\in_4 = \frac{\hbar^4}{8m_o^3 c^2} \tag{5}$$

It may be noted that the fourth order quantum mechanical energy wavefunction for the lsho U_4 is subject to the conditions of uniqueness and regularity everywhere and continuity across all boundaries and normalization.

It may be noted that the fourth order quantum mechanical energy wave equation for the lsho (3) may be written equivalently as

$$0 = [p_4(x) U_4''(x)]'' + [q_4(x) U_4(x)]'' + [\gamma_4 + r(x)] U_4(x) \tag{6}$$

where

$$p_4(x) = \in_4 \tag{7}$$

$$q_4(x) = \in_2 \tag{8}$$

$$\gamma_4 = E_4 \tag{9}$$

$$r_4(x) = -\frac{1}{2} m_o \omega_o^2 x^2 \qquad (10)$$

Consequently, that fourth order quantum mechanical energy wave equation for the SHO (3) and its conditions constitute a particular case of the fourth order Sturm-Liouville eigenproblem for which the existence of a complete set of exact eigensolutions has been established by F. B. Hildebrand[6]. Consequently, the exact analytic eigensolutions of the fourth order quantum mechanical wave equation for the lsho are guaranteed, a fortiori.

Now, precisely as in the solution of the second order quantum mechanical wave equation for the Isho, let us introduce the new independent variable defined by

$$\xi = a_4 x \qquad (11)$$

where a_4 is a constant parameter. Then (3) transforms as

$$0 = \left\{ \begin{array}{l} \in_4 a_4^4 U_4''''(\xi) + \in_2 a_4^2 U_4''(\xi) \\ + \left[E_4 - \dfrac{m_o \omega_{o2}}{2a_4^2} \xi^2 \right] U_4(\xi) \end{array} \right\} \qquad (12)$$

Next, precisely as in the solution of the second order quantum mechanical wave equation for the lsho above, let us seek the solution of (12) in the form

$$U_4(\xi) = \exp(-\lambda_4 \xi^2) F_4(\xi) \qquad (13)$$

where λ_4 is a constant parameter. Then the fourth order quantum mechanical energy wavefunction F_4 satisfies the equation

$$0 = \left\{ \begin{aligned} & \in_4 a_4{}^4 \{ F_4''''(\xi) - 8\lambda_4 \xi F_4'''(\xi) \\ & + \left[-12\lambda_4 + 24\lambda_4{}^2 \xi^2 \right] F_4''(\xi) \\ & + \left[48\lambda_4{}^2 - 32\lambda_4{}^3 \xi^3 \right] F_4'(\xi) \\ & + \left[12\lambda_4{}^2 - 48\lambda_4{}^3 \xi^2 + 16\lambda_4{}^4 \xi^4 \right] F_4(\xi) \} \\ & + \in_2 a_4{}^2 \{ F_4''(\xi) - 4\lambda_4 F_4'(\xi) + \left[-2\lambda_4 + 4_4{}^2 \xi^2 \right] F_4(\xi) \} \\ & + \left\{ E_4 - \frac{m_o \omega_o{}^2}{2a_4{}^2} \xi^2 \right\} F_4(\xi) \end{aligned} \right\} \qquad (14)$$

And dividing through by $\in_4 a_4{}^4$ and rearranging, (14) becomes

$$0 = \left\{ \begin{array}{l} F_4''''(\xi) - 8\lambda_4 \xi F_4'''(\xi) \\ + \left[(-12\lambda_4 + \alpha_4) + 24\lambda_4^{2}\xi^{2} \right] F_4''(\xi) \\ + \left[(48\lambda_4^{2} - 4\alpha_4\lambda_4) - 32\lambda_4^{3}\xi^{3} \right] F_4'(\xi) \\ \left[\begin{array}{l} \left[12\lambda_4^{2} - 2\alpha_4\lambda_4 + \dfrac{E_4 - m_o c^2}{\in_4 a_4^{4}} \right] \\ + \left[-48\lambda_4^{3} + 4\alpha_4\lambda_4^{2} - \dfrac{m_o\omega_o^{2}}{2 \in_4 a_4^{6}} \right] \xi^{2} \\ + 16\lambda_4^{4}\xi^{4} \end{array} \right] F_4(\xi) \end{array} \right\} \qquad (15)$$

where

$$\alpha_4 = \frac{\in_2}{\in_4 a_4^{2}} \qquad (16)$$

Next, precisely as in the solution of the second order quantum mechanical wave equation for the lsho, let us choose the parameter a_4 in the fourth order quantum mechanical energy wave equation (54) as

$$a_4 = \left(\frac{m_o \omega_o}{\hbar}\right)^{\frac{1}{2}} \tag{17}$$

Next, precisely as in the solution of the second order quantum mechanical wave equation for the lsho, let us choose the parameter λ_4 such that the coefficient of ξ^2 in the F_4 term in (15) vanishes:

$$0 = -48\lambda_4^{\,3} + 4\alpha_4\lambda_4^{\,2} - \frac{m_o \omega_o^{\,2}}{2 \in_4 a_4^{\,6}} \tag{18}$$

In the first place, it may be noted that equation (18) may be written equivalently as

$$0 = -48 \in_4 a_4^{\,6} \lambda_4^{\,3} + 4 \in_2 a_4^{\,4} \lambda_4^{\,2} - \frac{m_o \omega_o^{\,2}}{2} \tag{19}$$

Consequently, neglecting the term containing $\in_4$ as small compared with the others, it follows that

$$0 \cong 4 \in_2 a_4^{\,4} \lambda_4^{\,2} - \frac{m_o \omega_o^{\,2}}{2}$$

or

$$\lambda_4^{\,2} \cong \frac{m_o \omega_o^{\,2}}{8 \in_2 a_4^{\,4}} = \frac{1}{4}$$

Consequently,

$$\lambda_4 \cong \pm \frac{1}{2} \tag{20}$$

In the second place it may be noted that equation (18) may be written more conveniently as

$$0 = \left(\frac{1}{\lambda_4}\right)^3 + p\left(\frac{1}{\lambda_4}\right) + q \tag{21}$$

where

$$p = -4 \tag{22}$$

and

$$q = \frac{12\hbar\omega_o}{m_o c^2} \tag{23}$$

Consequently, equation (21) is a cubic equation in standard form with negative discriminant. Consequently, by the cubic formula its real solution is given by

$$\frac{1}{\lambda_4} = \left(\frac{-4p}{3}\right)^{\frac{1}{2}} \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{3q}{p(\frac{-4p}{3})^{\frac{1}{2}}}\right]\right\} \tag{24}$$

This relation gives the exact values of the parameter λ_4. And it may be shown **by** series expansion and a lot of manipulation that

$$\lambda_4 = \frac{1}{2}\left\{1 - \frac{3\hbar\omega_o}{4m_o c^2} - \frac{3^2\hbar^3\omega_o^{\ 2}}{4^2 m_o^{\ 2} c^4} \ldots\ldots\right\}^{-1} \tag{25}$$

Now with the above choices of a_4 and λ_4 our equation (54) becomes:

$$0 = \left\{ \begin{array}{l} F_4''''(\xi) - 8\lambda_4 \xi F_4'''(\xi) \\ + \left[(-12\lambda_4 + \alpha_4) - 24\lambda_4^2 \xi^2 \right] F_4''(\xi) \\ + \left[(48\lambda_4^2 - 4\alpha_4\lambda_4)\xi - 32\lambda_4^3 \xi^3 \right] F_4'(\xi) \\ + \left\{ \left[(12\lambda_4^2 - 2\alpha_4\lambda_4 + \frac{E_4}{\in_4 a_4^4} \right] + 16\lambda_4^4 \xi^4 \right\} F_4(\xi) \end{array} \right\} \tag{26}$$

Now let us seek the solution of (26) as a Frobenius series of the form

$$F_4(\xi) = \sum_{k=0}^{\infty} A_k \xi^{s+k} \tag{27}$$

where s is a constant index and A_k are constants. Then the indicial equation is given by

$$0 = s(s-1)(s-2)(s-3) \tag{28}$$

and hence

$$s = 0, 1, 2, 3 \tag{29}$$

And for each of these indices it follows that A_0, A_1, A_2, A_3 are arbitrary and

$$0 = \left\{ \begin{array}{l} 4!\, A_4 + \left[-4!\, \lambda_4 - 2!\, \alpha_4 \right] A_2 \\ + \left[12\lambda_4^2 - 2\alpha_4\lambda_4 + \frac{E_4}{\in_4 a_4^4} \right] A_0 \end{array} \right\} \tag{30}$$

and

$$0 = \begin{Bmatrix} 5!\,A_5 + \left[-5!\,\lambda_4 - - 3.2\alpha_4\right]A_3 \\ + \left[12.5\lambda_4^{\;2} - 6\alpha_4\lambda_4 + \dfrac{E_4}{\in_4\, a_4^{\;4}}\right]A_1 \end{Bmatrix} \qquad (31)$$

and

$$0 = \begin{Bmatrix} 6.5.4.3\,A_6 + \left[-7.4.4.3\lambda_4 - 4.3\alpha_4\right]A_4 \\ + \left[12.13\lambda_4^{\;2} - 10\alpha_4\lambda_4 + \dfrac{E_4}{\in_4\, a_4^{\;4}}\right]A_2 \end{Bmatrix} \qquad (32)$$

and

$$0 = \begin{Bmatrix} 7.6.5.4\,A_7 + \left[-9.5.4.4\lambda_4 - 5.4.\alpha_4\right]A_5 \\ + \left[12.25.\lambda_4^{\;2} - 14\alpha_4\lambda_4 - \dfrac{E_4}{\in_4\, a_4^{\;4}}\right]A_3 \\ -32\lambda_4^{\;3}A_1 \end{Bmatrix} \qquad (33)$$

and

$$0 = \left\{ \begin{array}{l} k(k-1)(k-2)(k-3)A_k \\ +[-4(k-2)(k-3)(2k-5)\lambda_4 -(k-2)(k-3)\alpha_4]A_{k-2} \\ +\left[\frac{E_4 - m_o c^2}{\in_4 a_4^4} - 2(2k-7)\alpha_4\lambda_4 + 12(2k^2-14k+25)\lambda_4^2\right]A_{k-4} \\ -32(k-6)\lambda_4^3 A_{k-6} \end{array} \right\} \quad (34)$$

The equations (30) - (34) constitute the recurrence relations for our fourth order quantum mechanical energy wave equation (26) for the Isho.

II. GROUND LEVEL OF THE FOURTH ORDER

For the ground level of the Isho to the fourth order let us choose the coefficient of A_0 in the recurrence relation for A_4 in (30) to vanish:

$$0 = \frac{E_4}{\in_4 a_4^4} - 2\alpha_4\lambda_4 + 12\lambda_4^2 \quad (35)$$

Then in the first place it follows from (35) and (5) and (16) and (17) that the fourth order quantum mechanical eigenenergy for the ground level of the fourth order of the Isho, denoted by $E_{4,0}$, is given by

$$E_{4.0} = \hbar\omega_o\lambda_4 - \frac{3\hbar^2\omega_o^2}{2m_oc^2}\lambda_4^2 \quad (36)$$

where λ_4 is given by (24). This is the exact fourth order quantum mechanical eigenenergy of the ground level of the Isho. It may be written more explicitly but approximately using (25) as

$$E_{4,0} = \begin{Bmatrix} +\frac{\hbar\omega_o}{2}\left\{1-\frac{3\hbar\omega_o}{4m_oc^2}-\frac{3^2\hbar^2\omega_o^{\ 2}}{4^2m_o^{\ 2}c^4}\cdots\cdots\right\}^{-1} \\ \\ -\frac{3\hbar^2\omega_o^{\ 2}}{8m_oc^2}\left\{1-\frac{3\hbar\omega_o}{4m_oc^2}-\frac{3^2\hbar^2\omega_o^{\ 2}}{4^2m_o^{\ 2}c^2}\cdots\right\}^{-2} \end{Bmatrix} \tag{37}$$

In the second place it follows from (35) and the recurrence relations (30) - (34) that for the ground level of the Isho the series solution of our fourth order quantum mechanical energy wave equation (26) corresponding to the arbitrary constant A_0, denoted by $F_{4,0,0}$ is given by

$$F_{4,0,0}(\xi) = \begin{Bmatrix} 1-\frac{16.4!.\lambda_4^{\ 4}}{8!}\xi^8 \\ \\ -\frac{16.4!.\lambda_4^{\ 4}}{10!}(\alpha_4+4.15.\lambda_4)\xi^{10} \\ \\ \cdots\cdots\cdots\cdots \end{Bmatrix} \tag{38}$$

In the third place it follows from (35) and the recurrence relations (30) - (34) that for the ground level of the Isho the series solution of the fourth order quantum mechanical energy wave equation (26) corresponding to the arbitrary constant A_1, denoted by $F_{4,0,1}$, is given by

$$F_{4,0,1}(\xi) = \begin{Bmatrix} \xi + \frac{4\lambda_4}{5!}(\alpha_4 - 12\lambda_4)\xi^5 \\ + \frac{1}{7!}[4\lambda_4(\alpha_4 + 9.4.\lambda_4)(\alpha_4 - 12\lambda_4) + 32.3!]\xi^7 \\ \dots\dots\dots \end{Bmatrix} \quad (39)$$

In the fourth place it follows from (35) and the recurrence relations (30) - (34) that for the ground level of the Isho the series of solution of the fourth order quantum mechanical energy wave equation (36) corresponding to the arbitrary constant A_2, denoted by $F_{4,0,2}$, is given by

$$F_{4,0,2}(\xi) = \begin{Bmatrix} \xi^2 + \frac{1}{4!}(2\alpha_4 + 4!\lambda_4)\xi^4 \\ + \frac{1}{6!}[16\lambda_4(\alpha_4 - 3.3.\lambda_4) - (\alpha_4 - 7.6.\lambda_4)(2\alpha_4 + 4!\lambda_4)]\xi^6 \\ \dots\dots\dots \end{Bmatrix} \quad (40)$$

In the fifth place, it follows from (35) and the recurrence relations (30) - (34) that for the ground level of the Isho the series of solution of the fourth order quantum mechanical energy wave equation (26) corresponding to the arbitrary constant A_3, denoted by $F_{4,0,3}$, is given by

$$F_{4,0,3}(\xi) = \begin{Bmatrix} \xi^3 + \frac{1}{5!}(3.2.\alpha_4 + 5!\lambda_4)\xi^5 \\ + \frac{1}{7!}[72\lambda_4(\alpha_4 - 24\lambda_4) + (\alpha_4 + 9.4.\lambda_4)(3.2.\alpha_4 + 5!\lambda_4)]\xi^7 \end{Bmatrix} \quad (41)$$

It is now obvious that the appropriate choice of our fourth order quantum mechanical eigenfunction for the ground level of the Isho is given by $F_{4,0,0}$. Consequently, for the ground level of the Isho the fourth order quantum mechanical energy eigenfunction, denoted by $U_{4,0}$, is given by

$$U_{4,0}(\xi) = N_{4,0}\exp(-\lambda_4\xi^2)F_{4,0,0}(\xi) \tag{42}$$

where λ_4 is given by (24) and $N_{4,0}$ is the corresponding normalization factor. Thus the solution of the fourth order quantum mechanical wave equation is complete for the ground level of the Isho.

III. FIRST LEVEL OF THE FOURTH ORDER

For the first level of the Isho let us choose the coefficient of A_1 in the recurrence relation for A_5 in (31) to vanish:

$$0 = \frac{E_4}{\in_4 a_4^4} - 6\alpha_4\lambda_4 + 12.5.\lambda_4^2 \tag{43}$$

Then it follows from (35) and (5) and (16) and (17) that the fourth order quantum mechanical eigenenergy for the first level of the Isho, denoted by $E_{4,1}$, is given by

$$E_{4,1} = 3\hbar\omega_o\lambda_4 - \frac{5.3.\hbar^2\omega_o^2}{2m_oc^2}\lambda_4^2 \tag{44}$$

where λ_4 is given by (34). This is the exact fourth order quantum mechanical eigenenergy of the first level of the Isho. It may be written more explicitly but approximately, using (25) as

$$E_{4,1} = \left\{ \begin{aligned} &+\frac{3\hbar\omega_o}{2}\left\{1-\frac{3\hbar\omega_o}{4m_oc^2}-\frac{3^2\hbar^2\omega_o^2}{4^2m_o^2c^4}\cdots\cdots\right\}^{-1} \\ &+\frac{5.3\hbar^2\omega_o^2}{8m_oc^2}\left\{1-\frac{3\hbar\omega_o}{4m_oc^2}-\frac{3^2\hbar^2\omega_o^2}{4^2m_o^2c^4}\cdots\cdots\right\}^{-2} \end{aligned} \right\} \tag{45}$$

Also, precisely as in the case of the ground level above, the fourth order quantum mechanical energy eigenfunction for the first level of the Isho follows from (43) and the recurrence relations (30) - (34).

IV. SECOND LEVEL OF THE FOURTH ORDER

For the second level of the Isho let us choose the coefficient of A_2 in the recurrence relation for A_6 in (32) to vanish:

$$0 = \frac{E_4}{\in_4 a_4^4} - 10\alpha_4\lambda_4 + 12.13.\lambda_4^2 \qquad \{46\}$$

Then it follows from (46) and (5) and (16) and (17) that the fourth order quantum mechanical eigenenergy for the second level of the Isho, denoted by $E_{4,2}$, is given by

$$E_{4,2} = 5\hbar\omega_o\lambda_4 - \frac{13.3.\hbar^2\omega_o^2}{2m_o c^2}\lambda_4^2 \qquad (47)$$

where λ_4 is given by (24). This is the exact fourth order quantum mechanical eigenenergy of the second level of the Isho. It may be written more explicitly but approximately, using (25), as

$$E_{4,2} = \left\{ \begin{array}{l} \dfrac{5\hbar\omega_o}{2}\left\{1 - \dfrac{3\hbar\omega_o}{4m_o c^2} - \dfrac{3^2\hbar^2\omega_o^2}{4^2 m_o^2 c^4}\cdots\cdots\right\}^{-1} \\ \\ + \dfrac{13.3\hbar^2\omega_o^2}{8m_o c^2}\left\{1 - \dfrac{3\hbar\omega_o}{4m_o c^2} - \dfrac{3^2\hbar^2\omega_o^2}{4^2 m_o^2 c^4}\cdots\cdots\right\}^{-2} \end{array} \right\} \qquad (48)$$

Also, precisely as in the case of the ground level above, the fourth order quantum mechanical energy wavefunction for the Isho follows from (29) and the recurrence relations (30) - (34).

V. THIRD LEVEL OF THE FOURTH ORDER

For the third level of the Isho let us choose the coefficient of A_3 in the recurrence relation for A_7 in (33) to vanish:

$$0 = \frac{E_4}{\in_4 a_4^4} - 14\alpha_4\lambda_4 + 25.12.\lambda_4^2 \qquad (49)$$

Then it follows from (49) and (5) and (16) and (17) that the fourth order quantum mechanical eigenenergy for the third level of the Isho, denoted by $E_{4,3}$, is given by

$$E_{4,3} = m_o c^2 + 7\hbar\omega_o\lambda_4 - \frac{25.3\hbar^2\omega_o^2}{2m_o c^2}\lambda_4^2 \qquad (50)$$

where λ_4 is given by (24). This is the exact fourth order quantum mechanical eigenenergy of the third level of the Isho. It may be written more explicitly but approximately, using (25), as

$$E_{4,3} = \left\{ \begin{array}{l} \frac{7\hbar\omega_o}{2}\left\{1 - \frac{3\hbar\omega_o}{4m_o c^2} - \frac{3^2\hbar^2\omega_o^2}{4^2 m_o^2 c^4}\cdots\right\}^{-1} \\ \\ + \frac{25.3\hbar^2\omega_o^2}{8m_o c^2}\left\{1 - \frac{3\hbar\omega_o}{4m_o c^2} - \frac{3^2\hbar^2\omega_o^2}{4^2 m_o^2 c^4}\cdots\right\}^{-2} \end{array} \right\} \qquad (51)$$

Also, precisely as in the case of the ground level above, the fourth order quantum mechanical eigenfunction for the third level of the Isho follows (49) and the recurrence relations (30)-(34).

VI. GENERAL LEVEL OF THE FOURTH ORDER

It is now obvious from the cases of the ground and first and second and third levels above, that in general the level of the Isho is obtained by choosing the coefficient of A_{k-4} in the recurrence for A_k in (34) to vanish:

$$0 = \left\{ \begin{array}{l} \frac{E_4}{\in_4 a_4^4} - 2(2k-7)\alpha_4\lambda_4 \\ \\ +12(2k^2 - 14k + 25)\lambda_4^2 \end{array} \right\} \; ; \quad k \geq 8 \qquad (52)$$

or equivalently,

$$0 = \left\{ \begin{array}{l} \dfrac{E_4}{\in_4 a_4^{\,4}} - 2(2n+1)\alpha_4 \lambda_4 \\ \\ +12(2n^2+2n+1)\lambda_4^{\,2} \end{array} \right\}; \quad n = 4,\ 5,\ 6 \qquad (53)$$

And it may be noted that this same relation correctly gives the eigenvalues for the ground level (n = 0) and first (n = 1) and second (n = 2) and third (n = 3) levels of the Isho as well. And it follows from (53) and (5) and (16) and (17) that the fourth order quantum mechanical eigenenergy for the nth level of the LSHO, denoted by $E_{4,n}$, is given by

$$E_{4,n} = \left\{ \begin{array}{l} (2n+1)\hbar\omega_o\lambda_4 \\ \\ \dfrac{3(2n^2+2n+1)\hbar^2\omega_o^{\,2}}{2m_o c^2}\lambda_4^{\,2} \end{array} \right\}; \quad = \ 0,\ 1,\ 2,\ldots\ldots \qquad (53)$$

where λ_4 is given by (34). This is the exact fourth order quantum mechanical eigenenergy of the nth level of the Isho. It may be written more explicitly but approximately, using (25) as

$$E_{4,n} = \left\{ \begin{array}{l} +(n+\tfrac{1}{2})\hbar\omega_o\left\{1 - \dfrac{3\hbar\omega_o}{4m_o c^2} - \dfrac{3^2\hbar^2\omega_o^{\,2}}{4^2 m_o^{\,2} c^4}\ldots\ldots\right\}^{-1} \\ \\ \dfrac{3(2n^2+2n+1)\hbar^2\omega_o^{\,2}}{8m_o c^2}\left\{1 - \dfrac{3\hbar\omega_o}{4m_o c^2} - \dfrac{3^2\hbar^2\omega_o^{\,2}}{4^2 m_o^{\,2} c^4}\ldots\ldots\right\}^{-2} \end{array} \right\}; \ . n = 0, 1, 2\ldots \qquad (55)$$

Also, precisely as in the case of the ground level above, the fourth order quantum mechanical energy eigenfunction for the general nth level of the lsho follows from (55) and the recurrence relations (30) - (34).

VII SUMMARY AND CONCLUSIONS

In this paper we derived the exact analytic eigensolutions of the kp^{2m} anharmonic scililationor correct to the order m = 2 in the case of the relativistic linear harmonic oscillator with hamitonian (1). The result is that there are infinitely many energy levels each of whose eigenenergy is precisely the well known Schrodinger's augmented by relativistic corrections of all orders of or c^{-2}. Consequently, the door is henceforth opened for the experimental investigation of the relativistic corrections to the eigenenergies of the harmonic oscillator. Also the way is cleared for the comparison of the exact eignenergies of the relativistic linear harmonic oscillator obtained in this paper with the approximate solutions obtained earlier by A. Maduemezia[4-5].

In this paper also we derived the recurrence relations for the exact analytic eigenfunctions of the relativistic linear harmonic oscillator. Therefore the door is henceforth opened for the investigation of the mathematical properties of these eigenfunctions and their comparison with the non-relativistic pure Schrodinger eigenfunctions: the Hermite's functions in particular and the hypergeometric functions in general.

Finally, the study in this paper paves the way for the investigation of the exact analytic eigensolutions of the kp^{2m} anharmonic oscillator correct to the order of all integral values of m.

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