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CLASSICALIZATION OF THE NONLINEAR SCHRODINGER EQUATION

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ABSTRACT

The Nonlinear Schrodinger Equation (NLSE), (in units of $\hbar = 1$)

$$i\frac{\partial \psi}{\partial t} + \alpha \frac{\partial^2 \psi}{\partial x^2} + \beta |\psi|^p \psi = 0$$

where $p = 1,2,3,\ldots;\alpha$ is a positive real constant, and β is a real constant, E is the energy, is known to be integral for infinite X space. A fourier analysis in a bounded interval $|X| \le L$, i.e.

$$u(x) = \sum_j u_j \exp\left(ik_j x\right)$$

shows that the NLSE is a nonlinear Hamiltonian system of N degrees of freedom. The effect of truncating the degrees of freedom to a finite number N and the fact that L is finite are investigated for the case $\alpha = +1, \beta = 1$ and $p = 2$. The results show that chaos sets in at certain value of the energy as the energy - increases for fixed N. Hence the NLSE is not integrable for finite N, and L. However, the integrability increases with increase in N implying that as N tends to infinity the NLSE becomes completely integrable as expected.

1. INTRODUCTION

The NLSE above describes the wave envelope $u(x)$ with time in a waveguide. The second term is the diffractive term, while the third is the nonlinear term with a nonlinear parameter β.

The analysis of this equation is important because its applications are ubiquitous since it is a canonical equation. It is a canonical equation because it governs the modulation of the amplitude u of a weakly nonlinear wave packet in a moving medium; (Drazin and Johnson 1992). It also finds an important application in Plasma Physics where it describes electron (Langmuir) waves. Another application is known in Nonlinear Optics where the function $u(x,t)$ has the sense of a complex envelope of electromagnetic field and the equation lf-modulation and self-focusing of light in a Kerr-type nonlinear ivshar Y.S. etal, 1989). It also has a geometric application in the e three-dimension motion of a vortex; (Rogers and Schief, 1998).

ork has been done for the linear Schrodinger equation (LSE) by 66. It was shown that the Hamiltonian operator of the LSE can be a sum of terms containing products of the q's (position coordinates) menta).

we express the NLSE as a Hamiltonian with infinity many degrees of ., as a sum of terms containing products of the q's and p's. In section ltonian equations of motion are formed from the Hamiltonian and tion 4 deals with the results and discussion. Section 5 concludes the

IEORETICAL ANALYSIS OF THE NONLINEAR NGER EQUATION:

$$i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{-\hbar^2}{2M} \nabla^2 + |\psi|^2 \right) \psi \tag{2.1}$$

onsiders a Hilbert space with basis $\phi_j(x)$ such that $\left\langle \phi_k \middle| \phi_j \right\rangle = \delta_{kj}$.

$$\psi = \sum_j u_j(t) \phi_j(x)$$

$$|\psi|^2 = \psi^* \psi = \sum_k \sum_l u_k^* u_l \phi_k^* \phi_l$$

eq (2.1) becomes

$$\sum_j \frac{\partial}{\partial t} u_j \phi_j = \left(\frac{-\hbar^2}{2M} \nabla^2 + \sum_k \sum_l u_k u_l \phi_k \phi_l \right) \sum_j u_j \phi_j$$

$$= \frac{-\hbar^2}{2M} \sum_j u_j \nabla^2 \phi_j + \sum_j \sum_k \sum_l u_j u_k^* u_l \phi_j \phi_k^* \phi_l$$

ng through by ϕ_m^* one gets

$$u_j \phi_m^* \phi_j = \frac{-\hbar^2}{2M} \sum_j u_j \phi_m^* \nabla^2 \phi_j + \sum_j \sum_k \sum_l u_j u_k^* u_l \phi_m^* \phi_j \phi_k \phi_l$$

$$\Rightarrow \frac{i\partial u_m}{\partial t} = \frac{-\hbar}{2M}\sum S_{mj}u_j + \frac{1}{\hbar}\sum_{jkl} u_j u^*_k u_l \qquad (2.2)$$

where

$$\phi^*_m \nabla^2 \phi_j = S_{mj}$$

and

$$\phi^*_m \phi_j \phi^*_k \phi_l = 1$$

Eq (2.2) is a coupled nonlinear equation of motion. It can be decoupled by using the transformation

$$u_n = p_n + iq_n$$

as follows:

$$i\left(\dot{p}_m + i\dot{q}_m\right) = \frac{-\hbar}{2M}\sum_j S_{mj}\left(p_j + iq_j\right) + \frac{1}{\hbar}\sum_{jkl}\left(p_j + iq_j\right)\left(p_k - iq_k\right)\left(p_l + iq_l\right)$$

$$= \frac{-\hbar}{2M}\sum_j S_{mj}\left(p_j + iq_j\right) + \frac{1}{\hbar}\sum_{jkl}\left(p_j p_k - ip_j q_k + ip_k q_j + q_j q_k\right)\left(p_l + iq_l\right)$$

$$= \frac{-\hbar}{2M}\sum_j S_{mj}\left(p_j + iq_j\right) + \frac{1}{\hbar}\sum_{jkl}\begin{pmatrix} p_j p_k p_l + ip_j p_k q_l - ip_j p_l p_k + p_j p_k p_l + ip_k p_l p_j - \\ p_k q_j q_l + p_l q_j q_k + iq_j q_k q_l \end{pmatrix}$$

Equating real and imaginary parts one obtains

$$\text{Im: } \dot{P}m = \frac{-\hbar}{2M}\sum_i S_{mj} q_j + \frac{1}{\hbar}\sum_{jkl}\left(p_j p_k q_l - p_j p_l q_k + p_k p_l q_j + q_j q_k q_l\right) = \frac{-\partial G}{\partial q_m}) \quad (2.3)$$

$$\text{Real: } \dot{q}_m = \frac{-\hbar}{2M}\sum_j S_{mj} p_j - \frac{1}{\hbar}\sum_{jkl}\left(p_j p_k q_l + p_j q_k q_l - p_k q_j q_l + p_l q_j q_k\right) = \frac{\partial G}{\partial p_m}) \quad (2.4)$$

where G is the Hamiltonian of the system.
From eq.(2.3)

$$G = \frac{\hbar}{2M}\sum_{jm} S_{mj} q_j q_m - \frac{1}{\hbar}\sum_{jklm}\left(p_j p_k q_l q_m - p_j p_l q_k q_m + p_k p_l q_j q_m + q_j q_k q_l q_m\right) \qquad (2.5)$$

From eq (2.4)

$$G = \frac{\hbar}{2M}\sum_{jm} S_{mj} p_j p_m - \frac{1}{\hbar}\sum_{jklm}\left(p_j p_k p_l p_m + p_j p_m q_k q_l - p_k p_m q_j q_l - p_l p_m q_j q_k\right) \quad (2.6)$$

By averaging equations (2.5) and (2.6) one gets

$$G = \frac{\hbar}{4M}\sum_{jm} S_{mj}\left(p_j p_m + q_j q_m\right) - \frac{1}{2\hbar}\sum_{jklm}\left(p_j p_k p_l p_m + p_j p_m q_k q_l - p_j p_m q_k q_l - p_j p_l q_k q_m\right.$$

$$\left. - p_k p_m q_j q_l + p_k p_l q_j q_m + p_l p_m q_j q_k + q_j q_k q_l q_m\right) \quad (2.7)$$

Eq(2.7) is a Hamiltonian with infinite degrees of freedom (d.o.f.)
One can truncate it to N degrees of freedom (d.o.f) to have

$$G = \frac{\hbar}{4M}\sum_{jm}^{N} S_{mj}\left(p_j p_m + q_j q_m\right) - \frac{1}{2\hbar}\sum_{jklm}\left(p_j p_k p_l p_m + p_j p_k q_l q_m + p_j p_m q_k q_l - p_j p_l q_k q_m\right.$$

$$\left. - p_k p_m q_j q_l + p_k p_l q_j q_m + p_l p_m q_j q_k + q_j q_k q_l q_m\right) \quad (2.8)$$

If N = 2 (i.e. 2 d.o.f) and ϕ's are eigenvectors of ∇^2 eq (2.8), **after expansion, simplification** and factorization becomes

$$G = \frac{\hbar}{4M}\left[S_{11}\left(p_1^2 + q_1^2\right) + S_{22}\left(p_2^2 + q_2^2\right) - \frac{1}{2\hbar}\left[p_1^2 + p_2^2 + q_1^2 + q_2^2 + 2(p_1 p_2 + q_1 q_2)\right]^2\right] \quad (2.9)$$

Similarly, for the case N = 3 Eq. (2.8) gives

$$G = \frac{\hbar}{4M}\left[S_{11}(p_1^2 + q_1^2) + S_{22}(p_2^2 + q_2^2) + S_{33}(p_3^2 + q_3^2)\right]$$

$$-\frac{1}{2}\left[p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2 + 2\left(p_1 p_2 + p_1 p_3 + p_2 p_3 + q_1 q_2 + q_1 q_3 + q_2 q_3\right)\right]^2 \quad (2.10)$$

(A.N.Njah, 2000).

3. THE EQUATIONS OF MOTION AND THEIR SOLUTIONS:

The Hamilton equations of motion for case N=2, from eq (2.9), are:

$$
\left.
\begin{aligned}
\dot{p}_1 &= \frac{-\partial G}{\partial q_1} = \frac{-\hbar}{2M} S_{11} q_1 + \frac{2}{\hbar}\left[p_1^2 + p_2^2 + q_1^2 + q_2^2 + 2\left(p_1 p_2 + q_1 q_2\right)\right]\left(p_1 + p_2\right) \\
\dot{p}_2 &= \frac{-\partial G}{\partial q_1} = \frac{-\hbar}{2M} S_{22} q_2 + \frac{2}{\hbar}\left[p_1^2 + p_2^2 + q_1^2 + q_2^2 + 2\left(p_1 p_2 + q_1 q_2\right)\right]\left(p_1 + p_2\right) \\
\dot{q}_1 &= \frac{\partial G}{\partial p_1} = \frac{\hbar}{2M} S_{11} p_1 - \frac{2}{\hbar}\left[p_1^2 + p_2^2 + q_1^2 + q_2^2 + 2\left(p_1 p_2 + q_1 q_2\right)\right]\left(q_1 + q_2\right) \\
\dot{q}_2 &= \frac{\partial G}{\partial p_2} = \frac{\hbar}{2M} S_{22} p_2 - \frac{2}{\hbar}\left[p_1^2 + p_2^2 + q_1^2 + q_2^2 + 2\left(p_1 p_2 + q_1 q_2\right)\right]\left(q_1 + q_2\right)
\end{aligned}
\right\} \quad (3.1)
$$

The system of first order differential equation eq (3.1) is linear in the first terms but nonlinear in the second (perturbation) term. By giving numerical values to S_{11}, S_{22}, M, and choosing appropriate initial conditions $q_i(0), p_i(0), i = 1,2$, the system can be solved numerically by the Runge-Kutta method. G can be calculated from eq(2.9). The phase diagram, i.e., the plots of points $(p_2, q_2), q_1 = 0$ *and* $p_1 \geq 0$ are shown in figs (1), and (2) and (3) for different values of S_{11}, S_{22} and hence E.

The equations of motion for the case N=3, from eq(2.10), are:

$$
\dot{p}_1 = \frac{-\partial G}{\partial q_1} = \frac{-S_{11} q_1}{2M} + 2\Big[p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2 + 2\left(p_1 p_2 + p_1 p_3 + p_2 p_3 + q_1 q_2 + q_1 q_3 + q_2 q_3\right)\Big]\left(q_1 + q_2 + q_3\right)
$$

$$
\dot{p}_2 = \frac{-\partial G}{\partial q_2} = \frac{-S_{22} q_2}{2M} + 2\Big[p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2 + 2\left(p_1 p_2 + p_1 p_3 + p_2 p_3 + q_1 q_2 + q_1 q_3 + q_2 q_3\right)\Big]\left(q_1 + q_2 + q_3\right)
$$

$$
\dot{p}_3 = \frac{-\partial G}{\partial q_3} = \frac{-S_{33} q_3}{2M} + 2\Big[p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2 + 2\left(p_1 p_2 + p_1 p_3 + p_2 p_3 + q_1 q_2 + q_1 q_3 + q_2 q_3\right)\Big]\left(q_1 + q_2 + q_3\right)
$$

$$\dot{q}_1 = \frac{\partial G}{\partial p_1} = \frac{S_{11}P_1}{2M} - 2\Big[p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2 + 2\left(p_1p_2 + p_1p_3 + p_2p_3 + q_1q_2 + q_1q_3 + q_2q_3\right)\Big]\left(p_1 + p_2 + p_3\right)$$

$$\dot{q}_2 = \frac{\partial G}{\partial p_2} = \frac{S_{22}q_2}{2M} - 2\Big[p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2 + 2\left(p_1p_2 + p_1p_3 + p_2p_3 + q_1q_2 + q_1q_3 + q_2q_3\right)\Big]\left(q_1 + q_2 + q_3\right)$$

$$\dot{q}_3 = \frac{\partial G}{\partial p_3} = \frac{S_{33}P_3}{2M} - 2\Big[p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2 + 2\left(p_1p_2 + p_1p_3 + p_2p_3 + q_1q_2 + q_1q_3 + q_2q_3\right)\Big]\left(q_1 + q_2 + q_3\right) \quad (3.2)$$

By giving numerical values to M, S_{ii}, q_i (0) and p_i (0), i = 1,2,3,… the system of equations in eq(3.2) can again be solved by the Runge-Kutta method. The results in phase space, are shown in figs (4), (5) and (6). G is now calculated from eq (2.10).

4. RESULTS AND DISCUSSION

For small values of E (E ≤ 1) the region, where the system is integrable is well demarcated by the ellipse. As E increases the region increases until chaos starts to set in at E = 1.7 for the case N = 2, fig. (2) and E = 5.2 for the case N = 3, fig. (5). As E increases further the system becomes chaotic and hence nonintegrable, fig. (3) and fig. (6) for the cases N = 2 and N = 3 respectively. It is also noted that, at the onset of chaos the region where the system is integrable is larger for the case N = 3 than the case N = 2, fig. (3.7). This shows that as N increases the region increases. In the limit that N tends to infinity, the region of integration tends to infinity and the NLSE is integrable for all values of E. In other words, for an infinitely large number of degree of freedom, the NLSE becomes completely integrable.

5. CONCLUSION

The NLSE, like the LSE, is also expressable as a sum of terms containing products of the q's (the position coordinates) and p's (the momenta). By this approach, it is found that the NLSE is completely integrable for an infinitely large number of degrees of freedom as expected.

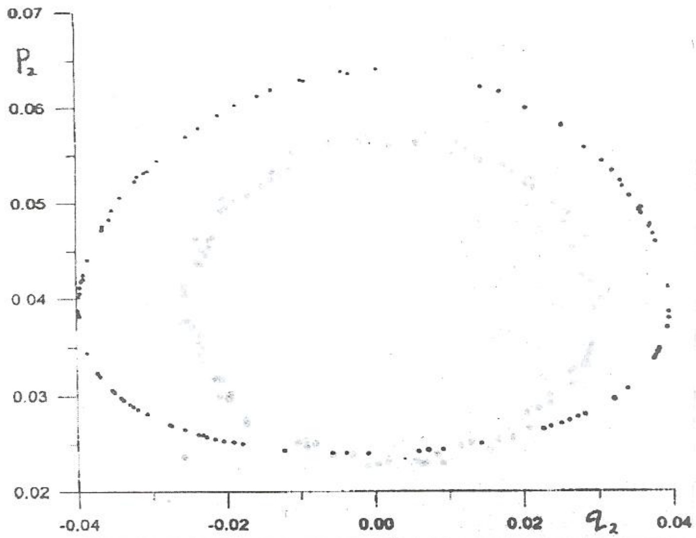

Fig (1): Phase space $\left(\rho_2 \, vs \, q_2\right)$ **of the NLSE, showing the KAM region**

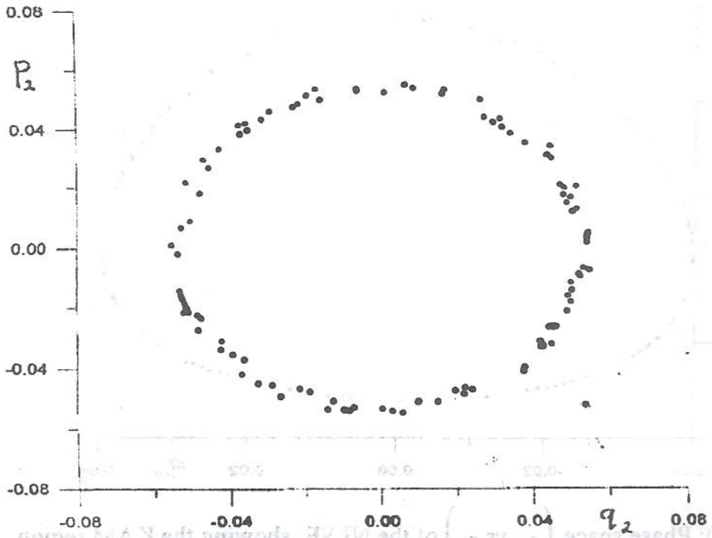

Fig (2): Phase space $(p2\ vs\ q2)$ **for the NLSE showing the onset of chaos**, case N=2.

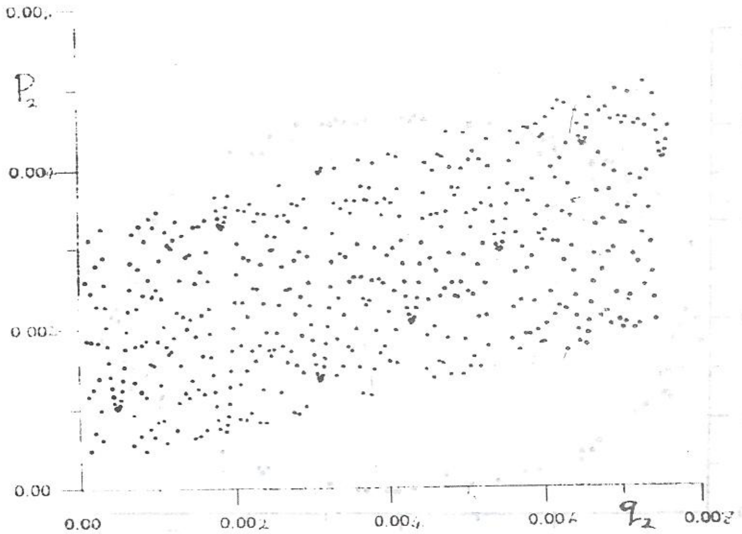

Fig (3): Phase space $\left(\rho_2 \, vs \, q_2\right)$ of the NLSE, case N=2 showing chaos.

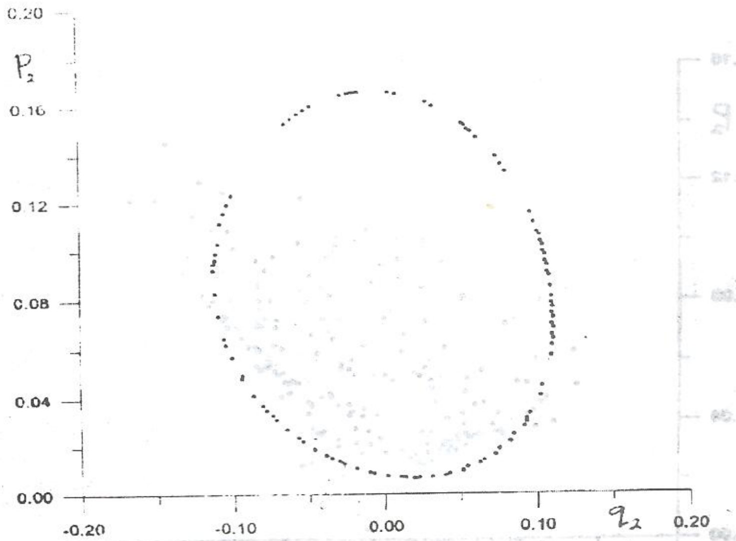

Fig (4): Phase space $\left(\rho_2 \, vs \, q_2\right)$ of the NLSE n=3 showing the KAM region

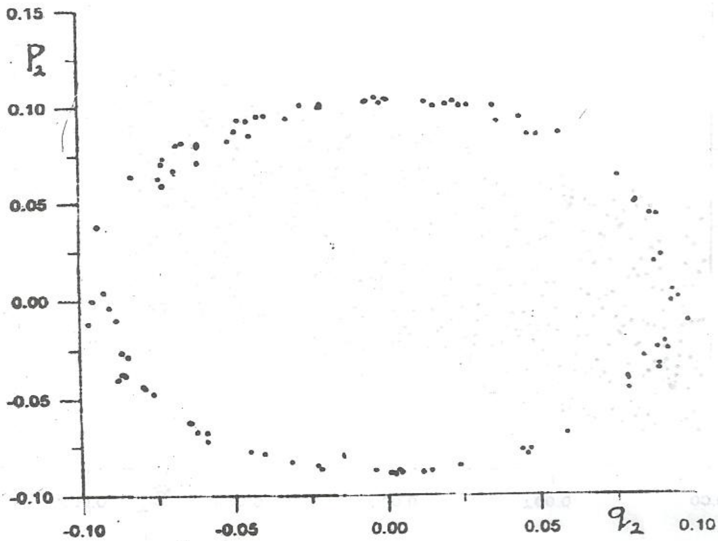

Fig (5): Phase space $\left(p2\ vs\ q2\right)$ for the NLSE, case N=3, showing the onset of chaos.

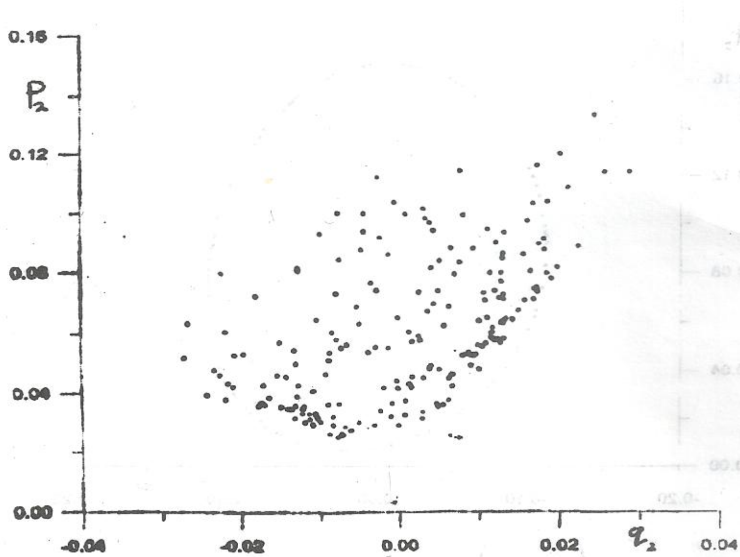

Fig (6): Phase space $\left(\rho_2\ vs\ q_2\right)$ of the NLSE with N=3, showing chaos.

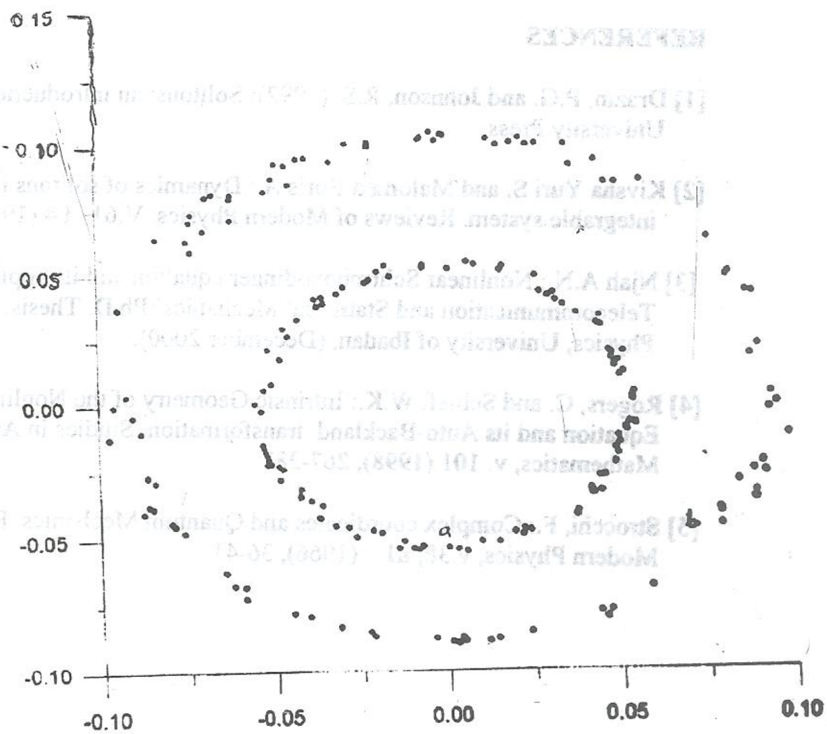

Fig (7): Phase space $\left(p2 \text{ vs } q2\right)$ comparing the KAM region at the onset of chaos for the cases (a) N=2 and (b) N=3 for the NLSE

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