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ERRATUM: SOME IDENTITIES IN SPINOR CALCULUS
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Section 2 of the published work should appear as follows:

2. THE IDENTITIES IN SPINOR CALCULUS

We shall now prove the following two identities:

$$\overline{\psi}'(x')\psi'(x') = \overline{\psi}(x)\psi(x) \tag{11}$$

$$\overline{\psi}'(x')\gamma^{\mu}\psi'(x') = \Lambda^{\mu}{}_{\nu}\overline{\psi}(x)\gamma^{\nu}\psi(x) \tag{12}$$

We shall start with the first identity. Let us note that ψ(x) is given by Eq. (10), and that in the Weyl (chiral) representation the γ-matrices are defined by

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix} \qquad \mu = 0,1,2,3 \tag{13}$$

From eqs. (10) and (13) we have with

$$\Psi(x) = \begin{pmatrix} \phi^{A} \\ \overline{\psi}^{\dot{A}} \end{pmatrix},$$

$$\overline{\Psi}(x) = \Psi^{+}(x)\gamma^{0} = \left(\phi^{A*}, \overline{\psi}^{*}_{\dot{A}}\right)\begin{pmatrix} 0 & \sigma^{0}_{A\dot{B}} \\ \overline{\sigma}^{0\dot{A}B} & 0 \end{pmatrix}$$

$$= \left(\overline{\psi}^{*}_{\dot{A}}\overline{\sigma}^{0\dot{A}B}, \phi^{A*}\sigma^{0}_{A\dot{B}}\right)$$

$$\therefore \overline{\Psi}(x)\Psi(x) = \overline{\psi}^{*}_{\dot{A}}\overline{\sigma}^{0\dot{A}B}\phi_{B} + \phi^{A*}\sigma^{0}_{A\dot{B}}\overline{\psi}^{\dot{B}} \tag{14}$$

We now use

$$\phi^{B} = \overline{\psi}^{*}_{\dot{A}}\overline{\sigma}^{0\dot{A}B} \tag{15}$$

Obtaining $\phi^{A}\sigma^{0}_{A\dot{C}} = \overline{\psi}^{*}_{\dot{B}}\overline{\sigma}^{0\dot{B}A}\sigma^{0}_{A\dot{C}} = \overline{\psi}^{*}_{\dot{B}}\delta^{\dot{B}}{}_{\dot{C}} = \overline{\psi}^{*}_{\dot{C}}$

i.e. $\overline{\psi}_{\dot{B}} = \phi^{A^{*}}\sigma^{0}_{A\dot{B}}$ (16)

Eq. (14) then becomes

$$\overline{\Psi}(x)\Psi(x) = \phi^{B}\phi_{B} + \overline{\psi}_{\dot{B}}\overline{\psi}^{\dot{B}} \tag{17}$$

Now $\Psi'(x') = \begin{pmatrix} \phi'_A \\ \overline{\psi}'^{\dot{A}} \end{pmatrix} = \begin{pmatrix} M_A{}^B \phi_B \\ \left(M^{*-1T}\right)^{\dot{A}}{}_{\dot{B}} \overline{\psi}^{\dot{B}} \end{pmatrix}$

Thus $\overline{\Psi}'(x') = \Psi'^{+}\gamma^0 = \left(\phi'^{A*}, \overline{\psi}'^{*}_{\dot{A}}\right) \begin{pmatrix} 0 & \sigma^0_{A\dot{C}} \\ \overline{\sigma}^{0\dot{A}C} & 0 \end{pmatrix}$

$= \left(\overline{\psi}'^{*}_{\dot{A}} \overline{\sigma}^{0\dot{A}C}, \phi'^{A*}\sigma^0_{A\dot{C}}\right) = \left(\phi'^{C}, \overline{\psi}'_{\dot{C}}\right) = \left(\phi'^{A}, \overline{\psi}'_{\dot{A}}\right)$

$$\therefore \overline{\Psi}'(x')\Psi'(x') = \left(\phi'^{A}, \overline{\psi}'_{\dot{A}}\right) \begin{pmatrix} \phi'_A \\ \overline{\psi}'^{\dot{A}} \end{pmatrix} = \phi'^{A}\phi'_A + \overline{\psi}'_{\dot{A}}\overline{\psi}'^{\dot{A}} \tag{18}$$

On using $\phi'^{A} = \left(M^{-1T}\right)^{A}{}_{B}\phi^{B}$ (19a)

$\overline{\psi}'^{\dot{A}} = \left(M^{*-1T}\right)^{\dot{A}}{}_{\dot{B}}\overline{\psi}^{\dot{B}}$ (19b)

Eq. (18) becomes

$\left(M^{-1T}\right)^{A}{}_{B}\phi^{B} M_A{}^C \phi_C + \left(M^{*}\right)_{\dot{A}}{}^{\dot{B}}\overline{\psi}_{\dot{B}} \left(M^{*-1T}\right)^{\dot{A}}{}_{\dot{C}}\overline{\psi}^{\dot{C}}$

$= \phi^{B}\left(M^{-1}\right)_{B}{}^{A} M_A{}^C \phi_C + \overline{\psi}_{\dot{B}}\left(M^{*T}\right)^{\dot{B}}{}_{\dot{A}}\left(M^{*-1T}\right)^{\dot{A}}{}_{\dot{C}}\overline{\psi}^{\dot{C}}$

$= \phi^{B}\delta^{C}_{B}\phi_C + \overline{\psi}_{\dot{B}}\delta^{\dot{B}}{}_{\dot{C}}\overline{\psi}^{\dot{C}} = \phi^{B}\phi_B + \overline{\psi}_{\dot{B}}\overline{\psi}^{\dot{B}}$

Which is the same as Eq. (17). This completes the proof of the first equality.
We now show that

$\overline{\psi}'(x')\gamma^{\mu}\psi'(x') = \Lambda^{\mu}{}_{\nu}\overline{\psi}(x)\gamma^{\nu}\psi(x)$

From Eq. (10)

$\overline{\Psi}(x) = \Psi^{+}(x)\gamma^0 = \left(\phi^{A*}, \overline{\psi}^{*}_{\dot{A}}\right) \begin{pmatrix} 0 & \sigma^0_{A\dot{A}} \\ \overline{\sigma}^{0\dot{A}A} & 0 \end{pmatrix}$

$= \left(\overline{\psi}^{*}_{\dot{A}}\overline{\sigma}^{0\dot{A}A}, \phi^{A*}\sigma^0_{A\dot{A}}\right) = \left(\phi^{A}, \overline{\psi}_{\dot{A}}\right)$

$\therefore \overline{\Psi}(x)\gamma^{\nu}\Psi(x) = \left(\phi^{A}, \overline{\psi}_{\dot{A}}\right) \begin{pmatrix} 0 & \sigma^{\nu}_{A\dot{B}} \\ \overline{\sigma}^{\nu\dot{A}B} & 0 \end{pmatrix} \begin{pmatrix} \phi_B \\ \overline{\psi}^{\dot{B}} \end{pmatrix}$

$= \left(\overline{\psi}_{\dot{A}}\overline{\sigma}^{\nu\dot{A}B}, \phi^{A}\sigma^{\nu}_{A\dot{B}}\right) \begin{pmatrix} \phi_B \\ \overline{\psi}^{\dot{B}} \end{pmatrix}$

$$= \overline{\psi}_{\dot{A}}\overline{\sigma}^{\nu\dot{A}B}\phi_B + \phi^{A}\sigma^{\nu}_{A\dot{B}}\overline{\psi}^{\dot{B}} \tag{20}$$

With $\Psi'(x') = \begin{pmatrix} \phi'_A \\ \overline{\psi}'^{\dot{A}} \end{pmatrix}$

$$\overline{\Psi}'(x') = \Psi'^{+}(x')\gamma^0 = \left(\phi'^{A*}, \overline{\psi}'^{*}_{\dot{A}}\right)\begin{pmatrix} 0 & \sigma^0_{A\dot{B}} \\ \overline{\sigma}^{0\dot{A}B} & 0 \end{pmatrix}$$

$$= \left(\overline{\psi}'^{*}_{\dot{A}}\overline{\sigma}^{0\dot{A}B}, \phi'^{A*}\sigma^0_{A\dot{B}}\right) = \left(\phi'^{B}, \overline{\psi}'_{\dot{B}}\right)$$

thus $\overline{\Psi}'(x')\gamma^{\mu}\Psi'(x') = \left(\phi'^{B}, \overline{\psi}'_{\dot{B}}\right)\begin{pmatrix} 0 & \sigma^{\mu}_{B\dot{B}} \\ \overline{\sigma}^{\mu\dot{B}B} & 0 \end{pmatrix}\begin{pmatrix} \phi'_B \\ \overline{\psi}'^{\dot{B}} \end{pmatrix}$

$$= \overline{\psi}'_{\dot{B}}\overline{\sigma}^{\mu\dot{B}B}\phi'_B + \phi'^{B}\sigma^{\mu}_{B\dot{B}}\overline{\psi}'^{\dot{B}}$$

$$= \overline{\psi}'_{\dot{A}}\overline{\sigma}^{\mu\dot{A}A}\phi'_A + \phi'^{A}\sigma^{\mu}_{A\dot{A}}\overline{\psi}'^{\dot{A}} \qquad (21)$$

we start by considering the first term of Eq. (21).

$$\overline{\psi}'_{\dot{A}}\overline{\sigma}^{\mu\dot{A}A}\phi'_A = \left(M^*\right)_{\dot{A}}^{\ \dot{B}}\overline{\psi}_{\dot{B}}\overline{\sigma}^{\mu\dot{A}A}M_A^{\ B}\phi_B$$

$$= \overline{\psi}_{\dot{B}}\left(M^+\right)^{\dot{B}}_{\dot{A}}\overline{\sigma}^{\mu\dot{A}A}M_A^B\phi_B$$

$$= \overline{\psi}_{\dot{C}}\delta^{\dot{C}}_{\dot{B}}\left(M^+\right)^{\dot{B}}_{\dot{A}}\overline{\sigma}^{\mu\dot{A}A}M_A^B\delta_B^C\phi_C \qquad (22)$$

we now use the identity

$$2\delta^{\dot{C}}_{\ \dot{B}}\delta_B^C = \overline{\sigma}^{\nu\dot{C}C}\sigma_{\nu B\dot{B}} \qquad (23)$$

Eq. (22) then becomes

$$\frac{1}{2}\overline{\psi}_{\dot{C}}\overline{\sigma}^{\nu\dot{C}C}\sigma_{\nu B\dot{B}}\left(M^+\right)^{\dot{B}}_{\dot{A}}\overline{\sigma}^{\mu\dot{A}A}M_A^B\phi_C$$

$$= \frac{1}{2}\left[\left(M^+\right)^{\dot{B}}_{\dot{A}}\overline{\sigma}^{\mu\dot{A}A}M_A^B\sigma_{\nu B\dot{B}}\overline{\sigma}^{\mu\dot{A}A}\right]\left[\overline{\psi}_{\dot{C}}\overline{\sigma}^{V\dot{C}C}\right]$$

$$= \frac{1}{2}Tr\left[M^+\overline{\sigma}^{\mu}M\sigma_{\nu}\right]\left[\overline{\psi}_{\dot{C}}\overline{\sigma}^{\nu\dot{C}C}\phi_C\right]$$

we now make use of the identity

$$\Lambda^{\mu}_{\nu} = \frac{1}{2}Tr\left[M^+\overline{\sigma}^{\mu}M\sigma_{\nu}\right] \qquad (24)$$

The first part of eq. (21) then becomes

$$\overline{\Psi}'_{\dot{A}}\overline{\sigma}^{\mu\dot{A}A}\phi'_{A} = \Lambda^{\mu}_{\nu}\overline{\Psi}_{\dot{A}}\overline{\sigma}^{\nu\dot{A}A}\phi_{A} \tag{25}$$

we now consider the second term in Eq. (21).
On using the relationship

$$\sigma^{\mu}_{\dot{A}A} = \varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\overline{\sigma}^{\mu\dot{B}B} \tag{26}$$

we have

$$\phi'^{A}\sigma^{\mu}_{A\dot{A}}\overline{\psi}'^{\dot{A}} = \phi'^{A}\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\overline{\sigma}^{\mu\dot{B}B}\overline{\psi}'^{\dot{A}}$$

(using Eqs. (19a) and (19b))

$$= \left(M^{-1T}\right)^{A}{}_{C}\phi^{C}\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\overline{\sigma}^{\mu\dot{B}B}\left(M^{*-1T}\right)^{\dot{A}}{}_{\dot{C}}\overline{\psi}^{\dot{C}}$$

$$= \left(M^{-1T}\right)^{A}{}_{C}\delta^{C}{}_{D}\phi^{D}\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\overline{\sigma}^{\mu\dot{B}B}\left(M^{*-1T}\right)^{\dot{A}}{}_{\dot{C}}\delta^{\dot{C}}{}_{\dot{D}}\overline{\psi}^{\dot{D}}$$

(on using eq. (23))

$$\frac{1}{2}\left(M^{-1T}\right)^{A}{}_{C}\overline{\sigma}^{\nu\dot{C}C}\sigma_{\nu D\dot{D}}\phi^{D}\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\overline{\sigma}^{\mu\dot{B}B}\left(M^{*-1T}\right)^{\dot{A}}{}_{\dot{C}}\overline{\psi}^{\dot{D}} \tag{27}$$

on using the identities

$$\left(M^{-1T}\right)^{A}{}_{C} = \varepsilon^{AE}M_{E}{}^{F}\varepsilon_{FC} \tag{28}$$

and

$$\left(M^{*-1T}\right)^{\dot{A}}{}_{\dot{C}} = \varepsilon^{\dot{A}\dot{E}}\left(M^{*}\right)_{\dot{E}}{}^{\dot{F}}\varepsilon_{\dot{F}\dot{C}} \tag{29}$$

Eq. (27) becomes

$$\frac{1}{2}M_{E}{}^{F}\varepsilon^{AE}\varepsilon_{FC}\overline{\sigma}^{\nu\dot{C}C}\sigma_{\nu D\dot{D}}\phi^{D}\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\overline{\sigma}^{\mu\dot{B}B}\left(M^{*}\right)_{\dot{E}}{}^{\dot{F}}\varepsilon^{\dot{A}\dot{E}}\varepsilon_{\dot{F}\dot{C}}\overline{\psi}^{\dot{D}} \tag{30}$$

upon using

$$\sigma^{\nu}_{F\dot{F}} = \varepsilon_{FC}\varepsilon_{\dot{F}\dot{C}}\overline{\sigma}^{\nu\dot{C}C} \tag{31}$$

Eq. (30) becomes

$$\frac{1}{2} M_E{}^F \varepsilon^{AE} \varepsilon^{\dot{A}\dot{E}} \sigma^{\nu}_{F\dot{F}} \sigma_{\nu D\dot{D}} \phi^D \varepsilon_{AB} \varepsilon_{\dot{A}\dot{B}} \overline{\sigma}^{\mu\dot{B}B} \left(M^*\right)_{\dot{E}}{}^{\dot{F}} \overline{\psi}^{\dot{D}} \tag{32}$$

Upon using

$$\varepsilon^{AE}\varepsilon_{AB} = (\varepsilon^T)^{EA}\varepsilon_{AB} = -\delta^E_B$$

and

$$\varepsilon^{\dot{A}\dot{E}}\varepsilon_{\dot{A}\dot{B}} = (\varepsilon^{\dot{T}})^{\dot{E}\dot{A}}\varepsilon_{\dot{A}\dot{B}} = -\delta^{\dot{E}}_{\dot{B}}$$

we have $\phi'^A \sigma^{\mu}_{A\dot{A}} \overline{\psi}'^{\dot{A}}$

$$= \frac{1}{2} M_B{}^F \sigma^{\nu}_{F\dot{F}} \sigma_{\nu D\dot{D}} \phi^D \overline{\sigma}^{\mu\dot{B}B} \left(M^*\right)_{\dot{B}}{}^{\dot{F}} \overline{\psi}^{\dot{D}}$$

$$= \frac{1}{2}\left[\overline{\sigma}^{\mu\dot{B}B} M_B{}^F \sigma_{\nu F\dot{F}} \left(M^+\right)^{\dot{F}}{}_{\dot{B}}\right]\left[\phi^D \sigma^V{}_{D\dot{D}} \overline{\psi}^{\dot{D}}\right]$$

on using eq. (24) we have

$$\phi'_A \sigma^{\mu}_{A\dot{A}} \overline{\psi}'^{A} = \Lambda^{\mu}{}_{\nu} \phi^D \sigma^{\nu}_{D\dot{D}} \overline{\psi}^{\dot{D}} \tag{33}$$

on combining eqs. (25) and (33) we obtain

$$\overline{\psi}'_{\dot{A}} \overline{\sigma}^{\mu\dot{A}A} \phi'_A + \phi'^A \sigma^{\mu}_{A\dot{A}} \overline{\psi}'^{\dot{A}}$$

$$= \Lambda^{\mu}{}_{\nu}\left(\overline{\psi}'_{\dot{B}} \overline{\sigma}^{\nu\dot{B}B} \phi_B + \phi^B \sigma^{\nu}_{B\dot{B}} \overline{\psi}^{\dot{B}}\right)$$

i.e., $\overline{\psi}'(x')\gamma^{\mu}\psi'(x') = \Lambda^{\mu}{}_{\nu}\overline{\psi}(x)\gamma^{\nu}\psi(x)$

This completes the proof of the equality.

REFERENCE

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