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ON THE APPLICATION OF LYAPUNOV'S THEOREM OF STABILITY TO THE CONVERGENCE OF THE REDUCED GRADIENT METHOD FOR OPTIMIZATION PROBLEMS WITH EQUALITY CONSTRAINTS

NJOSEH, I. N, OKONTA, P. N. & AJAYI, D. O.
Department Of Mathematics, Faculty Of Science,
Delta State University, Abraka

ABSTRACT

In this paper, we shall look at the convergence result of the reduced gradient method using the Cauchy problem for the system of ordinary differential equation. Lyapunov's theorem of stability will be applied to the convergence theorem to show that any numerical method that solves the Cauchy problem can also solve the problem of the form

$$\underset{x \in G}{Min}\, f(x)$$

where f(x) is the objective function, $G = \{x \in \Re^n : h(x) = 0\}$ and h(x) is the constraint function.

INTRODUCTION

Optimization might be defined as the science of determining the 'best' solution to certain mathematically defined problems, which are often models of physical reality. It involves the determination of algorithmic methods of solution, the study of the structure of such methods. In constrained optimization, the objective function is subject to some certain constraints (inequality or equality constraints).

Various authors using different methods have discussed both inequality and equality constrained optimization problems. These include Armijo [1], Fletcher [6], Rosen [13] etc. Most of these authors in discussing constrained optimization problems employed the Gradient, Conditional Gradient and the Reduced Gradient methods.

In this paper we shall review the work of Bertseka [2], Evtushenko [5] and Rosen [14] to ascertain that any numerical method that solves the given optimization problem (1.1) will also solve problem (1.0) with the aid of Lyapunov's theorem of Stability using the Reduced Gradient Method.

The following lemma and theorem will be applied in the course of this study.

Lemma 0.1: The projection onto $G = \{x : Ax = b\}$ is given by $P = I - A^T(AA^T)^{-1}A$.

Theorem 0.1: $\| y - A^T(AA^T)^{-1}Ay \|^2 = \|y\|^2 - \langle y, A^T(AA^T)^{-1}Ay \rangle$

THE REDUCED GRADIENT METHOD

For the solution of the problem of the form

$$\underset{x\in G}{Min} \quad f(x) \tag{1.0}$$

where f(x) is the objective function, $G = \{x \in \Re^n : h(x) = 0\}$ and h(x) is the constraint function, the reduced gradient method will be employed.

We shall consider the Cauchy problem for the system of ordinary differential equations.

$$\frac{dx}{dt} = -\nabla f(x) + \nabla h(x)^T[\nabla h(x)\nabla h(x)^T]^{-1}\nabla h(x)\nabla f(x) \tag{1.1}$$

Let $x_0 \in G$, then $h(x_0) = 0$.

If we let x(t) solve (1.1), the next series of the results will show that $x(t) \to x^*$ as $t \to \infty$, where x^* is the local solution of (1.0).

We require that G must be invariant with respect to (1.1) i.e., $x_0 \in G$ implies that

$x(t) \in G \ \forall \ t \geq 0$.

Equivalently,

$$\frac{dh(x(t))}{dt} = \nabla h(x(t))x'(t) \tag{1.2}$$

$$\equiv 0$$

Using (1.1)

$$\frac{df}{dt} = \nabla f(x)\left[-\nabla f(x) + \nabla h(x)^T\left(\nabla h(x)\nabla h(x)^T\right)^{-1}\nabla h(x)\nabla f(x)\right]$$

$$= -\|\nabla f(x)\|^2 + \left\langle \nabla f(x), \nabla h(x)^T\left[\nabla h(x)\nabla h(x)^T\right]^{-1}\nabla h(x)\nabla f(x)\right\rangle$$

$$= -\left[\|\nabla f(x)\|^2 - \left\langle \nabla f(x), \nabla h(x)^T\left[\nabla h(x)\nabla h(x)^T\right]^{-1}\nabla h(x)\nabla f(x)\right\rangle\right] \tag{1.3}$$

$$= -\left\|\nabla f(x) - \nabla h(x)^T\left[\nabla h(x)\nabla h(x)^T\right]^{-1}\nabla h(x)\nabla f(x)\right\|^2 \quad (by\ Theorem\ 0.1)$$

This shows that f(x(t)) monotonically decreases.

For the presentation of the proof of convergence of the reduced gradient method, we shall consider the proof of Lyapunov's asymptotic stability theorem.

The given system of ordinary differential equations that will be considered for the Lyapunov's asymptotic theorem is given as

$$\frac{dx}{dt} = H(x) \tag{1.4}$$

where H is continuously differentiable over $x \in \Re^n$.

If $H(x^*) = 0$, then the trivial solution $x = x^*$ of (1.4) is said to be asymptotically stable if, for any $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon)$ such that for any solution to (1.4) satisfying the condition

$$\|x_0\| < \delta,$$

we have

$$\begin{aligned} &\|x(x_0, t)\| \leq \varepsilon \quad \forall t \in \Re^n \\ &\lim_{t\to\infty} x(x_0, t) = x^* \end{aligned} \tag{1.5}$$

The solution $x \equiv x^*$ is said to be globally asymptotically stable if

i. it is Lyapunov stable

ii. for any $x_0 \in \Re^n$, (1.5) is satisfied.

We now represent the Lyapunov's theorem of stability.

THEOREM 1.1 (LYAPUNOV'S THEOREM OF STABILITY)

Let the point $x^* \in \Re^n$ be an equilibrium state for (1.4), i.e., $H(x^*) = 0$. Let $V: \Re^n \to \Re_+$ be defined and continuous together with its first derivative $^{dv}/_{dx}$ on an open set $G \in \Re^n$.

Define V(x) with respect to (1.4) by

$$V(x) = \nabla V(x).H(x)$$

then the continuous function V(x) is said to be positive definite with respect to x on G if $V > 0$ everywhere on G except the point x^*, where $V(x^*) = 0$. Similarly, if $V < 0$ everywhere on G except the point $x = x^*$, where $V(x) = 0$, we say that the function V(x) is negative definite with respect to x^* on G. If $V \geq 0$ ($V \leq 0$) holds everywhere on G, we say that the function V(x) on G is non-negative (non-positive) with respect to x^*.

We now state without proofs the Lyapunov's theorem on stability for ordinary differential equations.

THEOREM 1.2 (LYAPUNOV'S ASYMPTOTIC STABILITY THEOREM)

If there exists a differentiable function V(x) positive definite on G with V′(x) negative definite also on G, then the trivial solution $x = x^*$ of the system (1.4) is asymptotically stable.

APPLICATION TO THE CONVERGENCE OF THE REDUCED GRADIENT METHOD

THEOREM 1.3

Let the functions defining the problem (1.0) be differentiable on an open set containing $\Omega = \{x \in \Re^n : f(x) \le f(x_0),\ x \in G\}$ where $h(x) = 0$ satisfies the constraint qualification. Let the minimum of f(x) be attained on Ω at the unique point x*. Then the solutions of the system (1.1) converge to the point x* as $t \to \infty$.

PROOF

We use the Lyapunov's function

$$V(x) = f(x) - f(x^*).$$

Let x* be a strict relative minimum of f(x), then

$$V(x) > 0 \ \forall\ x \in G\backslash\{x^*\}$$

where

$$V(x^*) = f(x^*) - f(x^*) = 0.$$

Hence V(x) is positive definite.
We also see that

$$\begin{aligned} V'(x) &= \nabla V(x)\Big[-\nabla f(x) + \nabla h(x)^T\big[\nabla h(x)\nabla h(x)^T\big]^{-1}\nabla h(x)\nabla f(x)\Big] \\ &= \Big[\|\nabla f(x)\|^2 - \big\langle \nabla f(x), \nabla h(x)^T\big[\nabla h(x)\nabla h(x)^T\big]^{-1}\nabla h(x)\nabla f(x)\big\rangle\Big] \\ &= -\Big[\big\|\nabla f(x) - \nabla h(x)^T\big[\nabla h(x)\nabla h(x)^T\big]^{-1}\nabla h(x)\nabla f(x)\big\|^2\Big] \\ &< 0 \end{aligned}$$

Also V′(x*) = 0, showing that V(x) is negative definite on G.

Therefore, since V(x) is positive definite on G and V′(x) is negative definite also on G, the Lyapunov's theorem on asymptotic stability implies the local convergence of the method (1.1) to the point x*.
Thus any numerical method for solving (1.1) will also solve the problem (1.0).

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