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BONDED ELASTIC CYLINDRICAL SOLID UNDER ANTI-PLANE SHEAR

NNADI JAMES N.
Department Of Mathematics, Abia State University, Uturu, Nigeria

ABSTRACT

Longitudinal deformation fields within a long composite cylinder disturbed by anti-plane shear are investigated. The cylinder is made up of two semi-circular homogenous solids of different elastic moduli, perfectly bonded along their interface. The general form of the displacement and the stress states along the bond are deduced and depicted in graphical form.

1. INTRODUCTION

The utility of composite materials in construction industry requires the understanding of stress states within the material, when subjected to loads, to avoid failure especially along the bonds. Here, we study homogenous and isotropic semi-circular materials of elastic moduli μ_1 and μ_2 used to form a long solid cylinder perfectly bonded along their interface. Equal and opposite anti-plane shear of magnitude Q are applied along their lateral surface as depicted in fig. 1. The longitudinal displacement and stress distribution are then investigated. Different methods have been employed by various authors to analyze bimaterials. (see for example Erdogan [1], England [2], Zhang and Hasebe [3], Herrmann et al [4]). Here, the problem is analyzed by use of conformal mapping and Mellin transform.

2. BASIC NOTIONS

The character of anti-plane deformation is that the displacement components $w_i(r, \theta)$, i = 1, 2 in the z-direction are the only non-vanishing components of displacement which satisfy the equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)w_i(r,\theta)=0, 0\le r\le a, -\pi\le\theta\le\pi \tag{1}$$

Homogeneity of the interface require that

$$w_1(r,0^+)=w_2(r,0^-), w_1(r,\pi)=w_2(r,-\pi), 0\le r\le a \tag{2}$$

$$\sigma_{\theta z_1}(r,0^+)=\sigma_{\theta z_2}(r,0^-), \sigma_{\theta z_1}(r,\pi)=\sigma_{\theta z_2}(r,-\pi), 0\le r\le a \tag{3}$$

On the surface of the cylinder

$$\sigma_{\theta z_1}(a,\theta)=Q, 0\le\theta\le\pi; \quad \sigma_{\theta z_2}(a,\theta)=-Q, -\pi<\theta<0 \tag{4}$$

Polar stresses are related to displacement gradients through

$$\frac{\partial W_i}{\partial \theta}(r,\theta) = \frac{r}{\mu_i}\sigma_{\theta z_i}(r,\theta); \quad \frac{\partial W_i}{\partial r} = \frac{1}{\mu_i}\sigma_{r z_i}(r,\theta) \tag{5}$$

Where μ_i , i = 1, 2 are shear moduli and μ_0 is the modulus of the homogenous material.

3. TRANSFORMATION OF THE PROBLEM

The original plane of the analysis, $|z| \leq a$ id mapped onto the right half plane $\operatorname{Re}\zeta \geq 0$ by use of the holomorphic function

$$\zeta(z) = \frac{a+z}{a-z}, \quad z = x + iy \tag{6}$$

Setting $\zeta(z) = \rho e^{i\phi} = u(r,\theta) + iv(r,\theta)$ leads to

$$u(r,\theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2}; \; v(r,\theta) = \frac{2ar\sin\theta}{a^2 - 2ar\cos\theta + r^2}, \quad z = re^{i\theta}$$

Then

$$\rho(r,\theta) = \{u^2(r,\theta) + V^2(r,\theta)\}^{\frac{1}{2}} \tag{7}$$

$$\tan\phi(r,\theta) = \frac{2ar\sin\theta}{a^2 - r^2} \tag{8}$$

Therefore, for $-\pi \leq \theta \leq \pi$ and $r < a$

$$\frac{\partial \rho}{\partial r}(a,\theta) = 0; \; \frac{\partial \rho}{\partial \theta}(r,\theta) = 0; \quad \frac{\partial \rho}{\partial \theta}(r,\pm\pi) = 0 \tag{9}$$

$$\frac{\partial \phi}{\partial r}(a,\theta) = \frac{1}{a\sin\theta}; \; \frac{\partial \phi}{\partial \theta}(r,0) = \frac{2ar}{a^2 - r^2} = -\frac{\partial \phi}{\partial \theta}(r,\pm\pi) \tag{10}$$

Let the transform plane displacement be denoted by $W(\rho,\phi) \equiv w(r,\theta)$. Then, chain rule and (9a) give

$$\frac{\partial W_1}{\partial r}(a,\theta) = \frac{\partial W_1}{\partial \phi}\left(\rho,\tfrac{\pi}{2}\right)\frac{\partial \phi}{\partial r}(a,\theta) \quad 0 < \theta < \pi, \rho > 0 \tag{11a}$$

$$\frac{\partial W_2}{\partial r}(a,\theta) = \frac{\partial W_2}{\partial \phi}\left(\rho,-\tfrac{\pi}{2}\right)\frac{\partial \phi}{\partial r}(a,\theta) \quad -\pi < \theta < 0, \rho > 0 \tag{11b}$$

From (6) $z = a(\zeta - 1)(\zeta + 1)^{-1}$ so that on r = a

$$e^{i\theta} = \begin{cases} (i\rho-1)(i\rho+1)^{-1}, & \phi = \frac{\pi}{2}, \quad \rho > 0 \\ (i\rho+1)(i\rho-1)^{-1}, & \phi = -\frac{\pi}{2}, \quad \rho > 0 \end{cases}$$

Hence

$$a\sin\theta = \begin{cases} 2a\rho(1+\rho^2)^{-1} & 0<\theta<\pi \\ -2a\rho(1+\rho^2)^{-1} & -\pi<\theta<0 \end{cases} \tag{12}$$

Using (4), (5b) and (10a) – (12) we derive the boundary conditions

$$\frac{\partial W_1}{\partial\theta}(\rho,\tfrac{\pi}{2}) = \frac{2aQ\rho}{\mu_1(1+\rho^2)} \quad \rho>0 \tag{13}$$
$$\frac{\partial W_2}{\partial\theta}(\rho,-\tfrac{\pi}{2}) = \frac{2aQ\rho}{\mu_2(1+\rho^2)} \quad \rho>0$$

Continuity of displacement and (2) lead to

$$W_1(\rho,0) = W_2(\rho,0), \quad \rho>0 \tag{14}$$

Chain rule and (9b, c) give

$$\frac{\partial w_i}{\partial\theta}(r,\theta) = \frac{\partial W_i}{\partial\theta}(\rho,0)\frac{\partial\phi}{\partial\theta}(r,\theta), \quad 0<r<a, \begin{cases} \theta=0, \rho>1 \\ \theta=\pi, \rho<1 \end{cases}$$

Which together with (5a) and (10b) gives

$$\frac{\partial W_i}{\partial\phi}(\rho,0) = \frac{\partial W_2}{\partial\phi}(\rho,0), \quad \rho>0 \tag{15}$$

(The shear modulus has been considered to be μ_0).

Thus, in terms of $W_i(\rho,\phi)$, the problem is transformed into the task of solving for i = 1, 2

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)W_i(\rho,\phi) = 0, \ \rho>0, \quad -\frac{\pi}{2}\le\phi\le\frac{\pi}{2} \tag{16}$$

Subject to the boundary conditions (13) – (15).
Mellin integral transform denoted by

$$\hat{W}(s,\phi) = \int_0^{\infty} W(\rho,\phi)\rho^{s-1}d\rho$$

is then applied to (13) – (16) to get

$$\left(\frac{d^2}{d\phi^2} + s^2\right)\hat{W}_i(s,\phi) = 0, -1 < \mathrm{Re}\, s < 1, i = 1,2 \qquad (17)$$

$$\frac{\partial \hat{W}_i}{\partial \phi}(s, \pm\tfrac{\pi}{2}) = \frac{2aQ}{\mu_i} b(s)$$ (the negative sign is associated with i = 2)

where

$$b(s) = \frac{\pi}{2} \sec\frac{\pi}{2} s, \qquad = 3.2412 \qquad [5] \qquad (18)$$

$$\hat{W}_1(s,0) = \hat{W}_2(s,0); \ \frac{\partial \hat{W}_1}{\partial \phi}(s,0) = \frac{\partial \hat{W}_2}{\partial \phi}(s,0) \qquad (19)$$

The bounds in (17) we deduced from the asymptotic behaviours of the stresses as $\rho \to 0$ and as $\rho \to \infty$, which are obtained from (13)

$$\frac{\mu_i}{\rho}\frac{\partial \hat{W}_i}{\partial \phi}(\rho, \pm\tfrac{\pi}{2}) = \frac{2aQ}{1+\rho^2} \qquad \rho > 0$$

Thus if $W_i(\rho,\phi) = 0(\rho^k)$ then $\frac{\mu_i}{\rho}\frac{\partial W}{\partial \phi}(\rho,\phi) = o(\rho^{k-1})$

Hence k = 1 as $\rho \to 0$ and k = -1 as $\rho \to \infty$.
The solution of (17) may be written as

$$\hat{W}_i(s,\phi) = A_i(s)\sin s\phi + B_i(s)\cos\phi, \quad i = 1,2 \qquad (20)$$

Then (19a) gives

$$B_1(s) \quad B_2(s) \qquad (21)$$

while (19b) and

$$\frac{\partial B}{\partial \phi}(\rho,\phi) = sA_i(s)$$

gives

$$A_1(s) = A_2(s), \tag{22}$$

Using (18) and (20) we get

$$\begin{aligned} B_1(s) &= A_1(s)\frac{\cos\frac{\pi}{2}s}{\sin\frac{\pi}{2}s} - \frac{2aQb(s)}{\mu_1 s\sin\frac{\pi}{2}s} \\ B_2(s) &= -A_2(s)\frac{\cos\frac{\pi}{2}s}{\sin\frac{\pi}{2}s} - \frac{2aQb(s)}{\mu_2 s\sin\frac{\pi}{2}s} \end{aligned} \tag{23}$$

By (21),

$$B_1(s) - B_2(s) = 0$$

leads through (22) and (23) to

$$A_1(s) = \frac{a(\mu_1+\mu_2)Qb(s)}{\mu_1\mu_2 s\cos\frac{\pi}{2}s}$$

Substituting this value of $A_1(s)$ into (23) yields

$$B_1(s) = \frac{a(\mu_1-\mu_2)Qb(s)}{\mu_1\mu_2 s\cos\frac{\pi}{2}s}$$

With b(s) as given in (18b), write (20) as

$$\hat{W}(s,\phi) = \frac{\pi}{2}\frac{aQ}{\mu_1\mu_2}\left\{\frac{(\mu_1+\mu_2)}{s\cos^2\frac{\pi}{2}s}\sin s\phi + \frac{2(\mu_1-\mu_2)}{s\sin\pi s}\cos s\phi\right\} \tag{24}$$

The formula for the inverse Mellin transform is

$$W_k(\rho,\phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{W}_k(s,\phi)\rho^{-s}ds, \quad -1 < c < 1, k = 1,2$$

Which, relative to (24) gives the displacement as

$$W(\rho,\phi)=\frac{\pi aQ}{2\mu_1\mu_2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\frac{(\mu_1+\mu_2)}{s\cos^2\frac{\pi}{2}s}\sin s\phi+\frac{2(\mu_1-\mu_2)}{s\sin^2\pi s}\cos s\phi\right\}\rho^{-s}ds \quad (25)$$

The first term of the integrand in (25) has poles of order 2 at $s = \pm(2n-1)$, $n = 1, 2, 3 \ldots$ while the second term has one pole of order 2 at $s = 0$ and simple poles at $s = \pm n$, $n = 1,2,3, \ldots$ The integral is then evaluated by Cauchy's residue method and the result written according to Jordan's Lemma [6] which requires the closure of the contour in the right half planes $\text{Re}\,s > 0$ for $\rho > 1$ and closure of the contour in the left half plane $\text{Re}\,s < 0$ for $\rho < 1$. Since solutions are to be bounded we drop the solution arising from the pole at $s = 0$ and then write the result as the following uniformly convergent series:

$$W(\rho,\phi)=\frac{2aQ}{\pi\mu_1\mu_2}\begin{cases}(\mu_1+\mu_2)\left[-\phi\sum_{n=1}^{\infty}\frac{\rho^{2n-1}}{2n-1}\cos(2n-1)\phi+\sum_{n=1}^{\infty}\frac{\rho^{2n-1}}{(2n-1)^2}\sin(2n-1)\phi\right.\\ \left.-\ln\rho\sum_{n=1}^{\infty}\frac{\rho^{2n-1}}{2n-1}\sin(2n-1)\phi\right]-\frac{\pi}{2}(\mu_1-\mu_2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\rho\cos n\phi, & \rho<1\\ (\mu_1+\mu_2)\left[-\phi\sum_{n=1}^{\infty}\frac{\rho^{1-2n}}{2n-1}\cos(2n-1)\phi+\sum_{n=1}^{\infty}\frac{\rho^{1-2n}}{(2n-1)^2}\sin(2n-1)\phi\right.\\ \left.-\ln\rho\sum_{n=1}^{\infty}\frac{\rho^{1-2n}}{2n-1}\sin(2n-1)\phi\right]-\frac{\pi}{2}(\mu_1-\mu_2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\rho^{-n}\cos n\phi, & \rho<1\end{cases} \quad (26)$$

Substituting $\rho(r,\theta)=\left(\frac{a^2+2ar\cos\theta+r^2}{a^2-2ar\cos\theta+r^2}\right)^{\frac{1}{2}}$ (27)

And $\phi(r,\theta)=\tan^{-1}\left(\frac{2ar\sin\theta}{a^2-r^2}\right)$ (28)

Into (26) yields the displacement $w(r,\theta)$

4. STRESS STATES

The displacement gradients $\frac{\partial W}{\partial\phi}$ and $\frac{\partial W}{\partial\rho}$ may be written in terms of the following results:

For $0 < \rho < 1$, $\sum \rho^{2n}\cos(2n-1)\phi = \frac{\rho(1-\rho^2)\cos\phi}{1+2\rho^2\cos\phi+\rho^4}$;

$$\sum_{n=1}^{\infty}\rho^{2n-1}\sin(2n-1)\phi=\frac{\rho\left(1-\rho^{2}\right)\cos\phi}{1-2\rho^{2}\cos^{2}\phi+\rho^{4}};$$

$$\sum_{n=1}^{\infty}(-1)^{n}\rho^{n}\cos n\phi=\frac{-\rho(\rho+\cos\phi)}{1+2\rho\cos\phi+\rho^{2}};$$

$$\sum_{n=1}^{\infty}(-1)^{n}\rho^{n}\sin n\phi=\frac{-\rho\sin\phi}{1+2\rho\cos\phi+\rho^{2}};$$

These results are obtained following the procedure outlined in [7] by noting that for the first set we use

$$\sum_{n=1}^{\infty}z^{2n-1}=\frac{z}{1-z^{2}},\quad |z|<1$$

And for the other set, use is ma de of

$$\sum_{n=1}^{\infty}(-1)^{n}z^{n}=\frac{-z}{1+z},\quad |z|<1$$

Then setting $z=\rho e^{i\phi}$ with $0<\rho<1$ and comparing real and imaginary parts prove the results.

The expression for the gradients are:

$$\frac{\partial W}{\partial\phi}(\rho,\phi)=\frac{2aQ}{\pi\mu_1\mu_2}\left\{\frac{(\mu_1+\mu_2)}{1-2\rho^{2}\cos 2\phi+\rho^{4}}\left[\phi\rho\left(1+\rho^{2}\right)\sin\phi-\rho\ln\rho\left(1-\rho^{2}\right)\cos\phi\right]-\frac{\pi}{2}(\mu_1-\mu_2)\frac{\rho\sin\phi}{1+2\rho\cos\phi+\rho^{2}}\right\}\quad \rho<1 \tag{29a}$$

$$=\frac{2aQ}{\pi\mu_1\mu_2}\left\{\frac{(\mu_1+\mu_2)}{1-2\rho^{2}\cos 2\phi+\rho^{4}}\left[\phi\rho^{-1}\left(1+\rho^{-2}\right)\sin\phi-\rho^{-1}\ln\rho\left(1-\rho^{-2}\right)\cos\phi\right]-\frac{\pi}{2}\frac{(\mu_1-\mu_2)\rho^{-1}\sin\phi}{1+2\rho^{-1}\cos\phi+\rho^{-2}}\right\},\quad \rho<1 \tag{29b}$$

$$\frac{\partial W}{\partial\rho}(\rho,\phi)=\frac{2aQ}{\pi\mu_1\mu_2}\left\{\frac{(\mu_1+\mu_2)}{1-2\rho^{2}\cos 2\phi+\rho^{4}}\left[-\phi\left(1-\rho^{2}\right)\cos\phi-\ln\rho\left(1+\rho^{2}\right)\sin\phi\right]+\frac{\pi}{2}\frac{(\mu_1-\mu_2)(\rho+\cos\phi)}{1+2\rho\cos\phi+\rho^{2}}\right\}\quad \rho<1 \tag{30}$$

$$= \frac{2aQ}{\pi\mu_1\mu_2}\left\{\frac{(\mu_1+\mu_2)}{1-2\rho^{-2}\cos 2\phi+\rho^{-4}}\left[\phi\rho^{-2}\left(1-\rho^{-2}\right)\cos\phi-\rho^{-2}\left(1+\rho^{-2}\right)\sin\phi\right]\right. \qquad \rho>1$$
$$\left. -\frac{\pi}{2}\frac{(\mu_1-\mu_2)\rho^{-2}\left(\rho^{-1}+\cos\phi\right)}{1+2\rho^{-1}\cos\phi+\rho^{-2}}\right\}$$

Utilizing (5), chain rule, (29) and (30) the stresses everywhere in the bimaterial can be computed. The bond stresses are derived with the aid of the expressions

$$\frac{\partial w}{\partial\theta}(r,\theta)=\frac{\partial W}{\partial\phi}(\rho,0)\frac{\partial\phi}{\partial\theta}(r,\theta) \text{ and } \frac{\partial w}{\partial r}(r,\theta)=\frac{\partial W}{\partial\theta}(\rho,0)\frac{\partial\rho}{\partial r}(r,\theta) \qquad (31)$$

Where $\theta = 0$ or $\theta = \pi$, $0 < r < a$ and $r \to a$
We note that

$$\frac{\partial\rho}{\partial r}(r,\theta)=\frac{2a}{(a-r)^2}, \quad \theta=0, \quad 0<r<a$$
$$=\frac{-2a}{(a+r)^2}, \quad \theta=\pi, \quad 0<r<a$$

By (27) $\rho(r,\pi)=\rho^{-1}(r,0)=\frac{a-r}{a+r}$

Along the segment $0 < r < a$, $\theta = \pi$, ($\phi = 0$ and $\rho(r, \pi) < 1$), (10), (29a) and (31a) lead to

$$\sigma_{\theta z}(r,\pi)=\frac{\mu_0}{r}\frac{\partial W}{\partial\theta}(r,\pi)$$
$$=\frac{-\mu_0(\mu_1+\mu_2)}{\pi\mu_1\mu_2}Q\frac{a}{r}\ln\left(\frac{a+r}{a-r}\right) \qquad (32)$$

$$\sigma_{rz}(r,\pi)=\frac{\mu_0}{r}\frac{\partial W}{\partial r}(r,\pi)$$
$$=\frac{-\mu_0(\mu_1-\mu_2)}{\mu_1\mu_2}Q\frac{a}{a+r} \qquad (33)$$

Where μ_0 is the shear modulus of the homogenous material. At the corner $r = a$, $\theta = \pi$, ($\phi = 0$, $\rho(r,\pi)<1$), (29a) and (30a) are used when $\phi = 0$ and $\rho \to 0$ to obtain the asymptotic relations, as $\rho \to 0$ given by

$$\frac{\partial W}{\partial \phi}(\rho,0) = \frac{2a(\mu_1+\mu_2)}{\pi\mu_1\mu_2} Q\rho(r,\pi) \quad \text{and} \quad \frac{\partial W}{\partial \rho}(\rho,0) = \frac{a(\mu_1-\mu_2)Q}{\mu_1\mu_2}$$

Which lead to

$$\sigma_{\theta z}(a,\pi) = \lim_{r\to a} \sigma_{\theta z}(r,\pi) = \frac{-\mu_0(\mu_1+\mu_2)Q}{\pi\mu_1\mu_2} \tag{34}$$

$$\sigma_{rz}(a,\pi) = \lim_{r\to a} \sigma_{rz}(r,\pi) = \frac{-\mu_0(\mu_1-\mu_2)Q}{2\mu_1\mu_2} \tag{35}$$

A careful check shows that on the regiment $0 \le r < a$, $\theta = 0$, ($\phi = 0$, $\rho(r,0)>1$)

$$\sigma_{\theta z}(r,0) = \sigma_{\theta z}(r,\pi) \quad \text{and} \quad \sigma_{rz}(r,0) = \sigma_{rz}(r,\pi)$$

At the corner r = a, $\theta = 0$, ($\phi = 0$, $\rho(r,0)>1$) the equations associated with the asymptotic behaviours of (29b) and (30b) as $\rho\to\infty$ are applied to deduce:

$$\begin{aligned} \sigma_{\theta z}(0,0) &= \lim_{r\to 0} \frac{\mu_0}{r}\frac{\partial w}{\partial \theta}(r,0) \\ &= -\frac{2\mu_0(\mu_1+\mu_2)}{\pi\mu_1\mu_2} Q \lim_{r\to 0} \sum_{n=1}^{\infty} \frac{\left(\frac{r}{a}\right)^{2(n-1)}}{2n-1} \\ &= -\frac{2\mu_0(\mu_1+\mu_2)}{\pi\mu_1\mu_2} Q \end{aligned} \tag{36}$$

$$\sigma_{rz}(0,0) = \lim_{r\to 0} \frac{\mu_0}{r}\frac{\partial w}{\partial r}(r,0) = \frac{-\mu_0(\mu_1-\mu_2)}{\mu_1\mu_2} Q \tag{37}$$

Where μ_0 is the shear modulus of the homogenous bonded segment.

5. CONCLUSION

The longitudinal displacement along the bond is given by the nonlinear relation

$$\begin{aligned} W(r,0) &= \frac{\mu_1-\mu_2}{\mu_1\mu_2} aQ \ln\left(\frac{2a}{a+r}\right), \quad 0 \le r < a, \ \theta = 0 \ \text{or}\ \theta = \pi \\ &= 0, \qquad r = a \quad \left(\rho(a,\pi) \to 0\right) \end{aligned}$$

This character of the displacement disappears if the cylinder is homogenous, for in that case $\mu_1 = \mu_2 = \mu_0$ and our results agree with those of [8].

The presence of $\sigma_{rz}(r, \pi)$, $0 < r < a$ implies the introduction of a surface that would have been absent had the cylinder been homogenous. Cracks may initiate at the edges $r = a$ $\theta = 0$ and $\theta = \pi$ which experience higher intensity of $\sigma_{\theta z}(r,\theta)$ as $r \to a$. The relationship between the magnitudes of

$$T(r,\pi) = \frac{\mu_1\mu_2}{\mu_0(\mu_1+\mu_2)}\frac{\sigma_{r\theta}}{Q}(r,\pi) \quad \text{and} \quad \hat{T}(r,\pi) = \frac{\mu_1\mu_2}{\mu_0(\mu_1-\mu_2)}\frac{\sigma_{rz}}{Q}(r,\pi) \quad \text{against} \quad \frac{r}{a}$$

are shown in graphs.

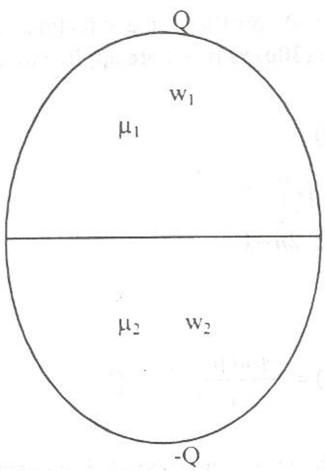
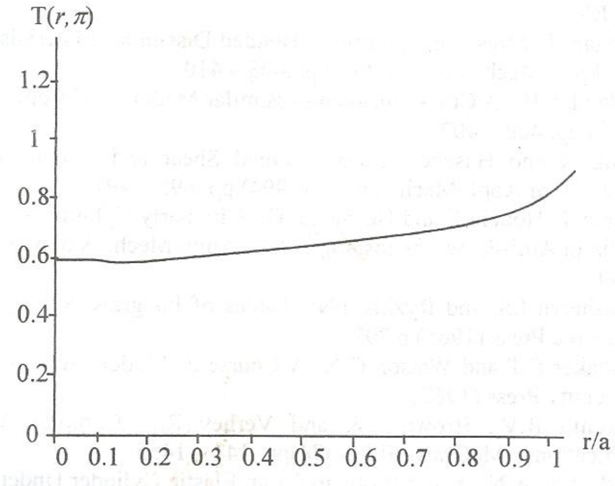
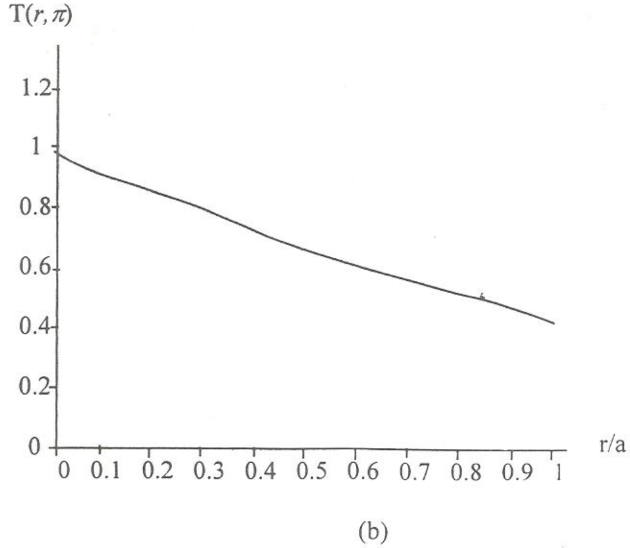


Fig 1 Bonded semicircular materials subjected to Anti-plane shear



(a)



(b)

Fig II (a) Variation of $T(r,\pi) = \dfrac{\mu_1\mu_2}{\mu_0(\mu_1+\mu_2)}\sigma_{\frac{\theta z}{\varrho}}$ with $\dfrac{r}{a}$

(b) Variation of $\hat{T}(r,\pi) = \dfrac{\mu_1\mu_2}{\mu_0(\mu_1-\mu_2)}\sigma_{\frac{\theta z}{\varrho}}$ with $\dfrac{r}{a}$

REFERENCES

[1]. Erdogan, F. Stress Distribution in Bonded Dissimilar Materials with Cracks. J. of Appl. Mech. Vol 32 (1965) pp.403 – 410.

[2]. England A.H.: A Crack Between Dissimilar Media. J. of Appl. Mech. Vol 32 (1965) pp. 400 – 402.

[3]. Zhang X and Hasebe N: Longitudinal Shear and Torsion of Bimaterial Solids. J. of Appl. Mech. Vol 61 (1994) pp. 495 – 497.

[4]. Honein T, Honein E and Herrmann G: Circularly Cylindrical Plane Layered Media in Anti-Plane Elastostatics. J. of Appl. Mech. Vol 61 (1994) pp. 243 – 250.

[5]. Gradshteyn I.S. and Ryzhik I.N:. Tables of Integrals, Series and Products. Academic Press (1965),p 292.

[6]. Whittaker E.T and Watson G.N. A Course of Modern Analysis. Cambridge University Press (1962).

[7]. Churchill R.V., Brown J.W and Verhey R.F. Complex Variables and Applications. McGraw-Hill (1974)pp. 144 – 1455.

[8]. Nnadi, J.N. A Neumann Problem for an Elastic Cylinder Under Out-of-plane Load. J. of Pure and Appl. Sci. Vol 7 No. 3 (2001) (To appear)