J. Nig. Ass. Math. Phys.
Vol. 5 (2001)

EXISTENCE AND UNIQUENESS FOR SELF-SIMILAR SOLUTIONS FOR A VISCOUS REACTING FLOW

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ABSTRACT

In this paper, we examine a problem of the form;

$$\frac{\partial U}{\partial t} = \sum_{i=1}^{N} \frac{1}{r^m} \frac{\partial}{\partial x_i}\left(r^m \frac{\partial U}{\partial x_i} \right) + F(x,t,U)_i$$

$x \in R^n$, $t > 0$ and

$$r = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

which represents a viscous reacting flow problem. We provide an example and methods of solving the problem. In particular, we are interested in a self-similar solution. The conditions for the existence and uniqueness of self-similar solution are investigated.

NOMENCLATURE.

ρ	-	density
v	-	velocity
p	-	pressure
T	-	Temperature
Y_f	-	mass function of the fuel
Y_o	-	mass function of the oxygen
Y_p	-	mass function of the product.
ε_o	-	reactivation energy
Y	-	pre-mixed reactants
C	-	specific heat
t	-	time
R	-	universal gas constant
D	-	diffusion coefficient
K	-	thermal conductivity

1 INTRODUCTION

The mathematical models of viscous reacting flows with both small and large activation energies are presented. We examine the self-similar solutions of these models. The models represent combustion problems in a tube, which simplifies the reacting system.

Buckmaster & Ludford [4] examined the work of Fran-Kamenelslii done forty years earlier and introduced approximation based on large activation energy to construct a thermal theory of spontaneous combustion. In combustion, reactions are encountered which take place in the gas phase between a fuel and an oxidizer. Chorin [5] observed that, in combustion problems, time-dependent phenomena occur in a three-dimensional space and the number of equations to be solved is often substantial. Besides, there are several distinct length and time scales to be resolved and the flow in which the combustion occurs is usually turbulent. The combustion reactions are exothermic and the reaction rates are very sensitively dependent on temperature. Often, the fuel is initially in solid or liquid form. These fuels must gasify before mixing and chemical reaction with the oxygen may take place in the form of jets that break or atomise, to form a spray of small droplets that vaporize to generate the fuel vapours that mix and react with ambient air. According to Ajadi [1], the reaction is modeled by an overall reaction of the form,

$$F + nO_2 \rightarrow (1+n)p + (q) \tag{1.1}$$

where a mass n of oxygen is consumed per unit mass of fuel to produce a mass (1+n) of product and thermal energy (q).

2 MATHEMATICAL EQUATIONS

The mathematical equations governing the above reaction are continuity, momentum, species and energy equations given below.

$$\frac{\partial U}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v^2) = \rho g - \nabla \cdot p + \nabla \cdot r \tag{2.2}$$

$$\frac{\partial}{\partial t}(\rho Y_a) + \nabla \cdot (\rho v Y_a) = \nabla \cdot (\rho D_a \nabla Y_a) + w_a \tag{2.3}$$

$$\frac{\partial}{\partial t}(\rho C_p T) + \nabla \cdot (\rho v C_p T) = \nabla \cdot (k \nabla T) - q w_F \tag{2.4}$$

where, v, ρ, p, Y_a, w_a ,g and T are defined accordingly. Also, Ajadi [1] observed that if velocity is zero, then density is constant, hence,

$$w_F = -Y_0 Y_F B(T) e^{\varepsilon_0/RT} \quad . \tag{2.5}$$

where ε_0 is the activation energy, and R is the universal gas constant. If Y_α is constant, equation (2.5) reduces to,

$$w_F = -\lambda T^m e^{\varepsilon_0/RT} \tag{2.6}$$

Since ε_o determines the nature of the reaction, he considered $\varepsilon_o = 0$, which reduces equation (2.4) to,

$$\rho C_p \frac{\partial T}{\partial t} = \nabla \cdot (k(T)\nabla T)\lambda T^m \tag{2.7}$$

But, in this paper, we still consider the problem when activation energy is small, that is, $\varepsilon_o = 0$, but in subsequent cases the problem shall be examined at $\varepsilon_o \neq 0$.

In general, we consider a problem of the form,

$$\frac{\partial T}{\partial t} = \sum_{i=1}^{N} \left[\frac{1}{r^m} \frac{\partial}{\partial x_i} \left(r^m \frac{\partial T}{\partial x_i} \right) \right] + F(x,t,T)_i$$

$x \in R^n$, $t > 0$ and $r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

(2.8) $\qquad T(x,0) = T_o(x)$

$$\lim_{x \to \pm\infty} T(x,t) = 0$$

This equation arises in many areas of applications, such as heat flow in materials with a temperature dependent conductivity with or without reaction. The function F depends on the reaction. For example, according to Ayeni [3], the gas phase reaction;

$$\tfrac{1}{2}N_2 + \tfrac{1}{2}H_2 = NH_3 \tag{2.9}$$

that is,

$$v_1 H + v_2 H_2 + v_3 NH_3 = -\tfrac{1}{2}N_2 - \tfrac{1}{2}H_2 + NH_3 \tag{2.10}$$

has

$$F(T) = exp\left(-\sum (v/R)\right) \int (v_1 - Rl/PC)\, dp \tag{2.11}$$

From equation (2.8) many equations can be formulated depending on the reaction as mentioned above. In this paper, we shall consider only case 1, in which

$$F(x,\ t,\ u) = (t - t_0)^{\alpha} T^{n} \tag{2.12}$$

Putting equation (2.12) into equation (2.8), with N=1 and $x_1 = x$, we obtain,

$$\frac{\partial T}{\partial t} = \frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m \frac{\partial T}{\partial x}\right) + (t - t_0)^{\alpha} T^{n}$$

with initial condition

$$T(x,0) = 1$$

and boundary conditions

$$\begin{aligned} T(0,t) &= 1 \\ T(\infty,t) &= 0,\ t > t_o \end{aligned} \tag{2.13}$$

We use shooting method to transform equation (2.13) to,

$$\frac{\partial T}{\partial t} = \frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m \frac{\partial T}{\partial x}\right) + (t - t_0)^{\alpha} T^{n}$$

with initial condition

$$\begin{aligned} T(0,t) &= 1 \\ T'(0,t) &= -\gamma,\ \ t > t_o,\ \gamma > 0,\ \ t > t_o \end{aligned} \tag{2.14}$$

where $\gamma > 0$ and it is to be determined such that the boundary conditions are satisfied.

3. SELF-SIMILAR SOLUTIONS

The similarity solution method is used for reducing a (nonlinear) partial differential equation to an ordinary differential equation. Thus, a problem involving two independent variables or more is reduced to one. This approach was used in the linear case to solve a Stefan problem and the famous Navier-Stokes equation, each giving rise to either a closed form analytic solution or a numerical solution.
We seek a self-similar solution of the form,

$$\eta = x^p (t - t_o)^q,\ \ T(x,t) = f(\eta) \tag{3.1}$$

Now, simplifying equation (2.13), we obtain

$$\frac{\partial T}{\partial t} = \frac{m}{x}\frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2} + (t-t_0)^{\alpha} T^{n} \tag{3.2}$$

and taking into account equation (3.1), we get

$$\frac{\partial T}{\partial t} = qx^{p}(t-t_0)^{q-1}\frac{df}{d\eta} \tag{3.3}$$

similarly,

$$\frac{\partial T}{\partial t} = px^{p-1}(t-t_0)^{q}\frac{df}{d\eta} \tag{3.4}$$

and

$$\frac{\partial T}{\partial t} = p^2 x^{2p-2}(t-t_0)^{2q}\frac{df}{d\eta} \tag{3.5}$$

substituting equations (3.1), (3.3), (3.4) and (3.5) into equation (3.2) and simplifying, we obtain,

$$\frac{\partial^2 f}{\partial\eta^2} = \left[\frac{q\eta^{2/p}(t-t_0)^{-(p+q)/p} - mp}{p^2\eta}\right]\frac{df}{d\eta} - \left[\frac{\eta^{2/p-2}(t-t_0)^{(\alpha p+q)/p}}{p^2}\right]f^{n} \tag{3.6}$$

Setting

$$\begin{aligned} -p-2q &= 0 \\ \alpha p - 2q &= 0 \end{aligned} \tag{3.7}$$

Equation (3.6) can be written as,

$$\frac{d^2 f}{d\eta^2} = \left[\frac{q\eta^{2/p} - mp}{p^2\eta}\right]\frac{df}{d\eta} - \frac{\eta^{2/p-2}f^{n}}{p^2} \tag{3.8}$$

Important Remark: Equation (3.8) is the requirement for similarity solution. Equation (3.8) can be written as

$$f'' = \left[\frac{q\eta^{2/p} - mp}{p^2 \eta} \right] f' - \frac{\eta^{2/p - 2} f^n}{p^2}$$

$$f(0) = 1, \quad f(\infty) = 0 \tag{3.9}$$

As a result of the difficulty involve in solving the boundary value problem in equation (3.9) when the finite-difference method is used, therefore (3.9) is transformed to initial value problem by means of shooting method as discussed in Alao [2].
Hence equation (3.9) becomes,

$$f'' = \left[\frac{q\eta^{2/p} - mp}{p^2 \eta} \right] f' - \frac{\eta^{2/p - 2} f^n}{p^2}$$

$$f(0) = 1, \quad f'(0) = -\gamma \tag{3.10}$$

where γ is unknown and to be determined, such that the boundary conditions are satisfied. That is γ is to be guessed until the approximated solution is equivalent to exact solution. Numerical solution of this problem in Alao [2] shows $\gamma = 1$. Equation (3.10) is changed into systems of linear equation as follows,

Let

$$x_1 = \eta$$
$$x_2 = f$$
$$x_3 = f'$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ \left(\frac{q x_1^{2/p} - mp}{p^2 x_1} \right) x_3 - \frac{x_1^{2/p - 2} x_2^{\,n}}{p^2} \end{pmatrix} \tag{3.11}$$

with the conditions.

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -\gamma \end{pmatrix} \qquad (3.12)$$

We state existence and uniqueness results as follows;
Let $1 < x_1 < \infty$, $1 < x_2 < \infty$ and $-1 < x_3 < 0$. Also, for every $n \leq 0$, $2 \leq p < \infty$ and $-\infty < x_3 \leq -1$, then, there exists a unique solution of the problem (3.9).
We sketch the proof as below;
From (3.11, we have)

$$\begin{aligned} f_1 \to f_1(x_1, x_2, x_3) &= 1 \\ f_2 \to f_2(x_1, x_2, x_3) &= x_3 \\ f_3 \to f3(x_1, x_2, x_3) &= \left(\frac{q x_1^{2/p} - mp}{p^2 x_1} \right) x_3 - \frac{x_1^{2/p-2} x_2^{\,n}}{p^2} \end{aligned}$$

By the existence and uniqueness theorem, the boundedness of $\left| \frac{\partial f_i}{\partial f_j} \right|$, for all i,j, implies the existence of unique solution [6]. Therefore, when the boundedness of (3.1) was investigated by Alao [2], it was discovered that $\left| \frac{\partial f_i}{\partial f_j} \right|$, for i,j = 1,2,3 are bounded. Thus, the theorem follows.

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