J. Nig. Ass. Math. Phys.
Vol. 5 (2001)

THE USE OF THE GAUSS-LEGENDRE QUADRATURE IN SOLVING FLOW PROBLEMS IN RESERVOIRS CONTAINING HORIZONTAL WELLS

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ABSTRACT

In horizontal wells reservoir fluid flow is governed in behaviour by well geometry and reservoir properties in such a manner that none of the existing methods can be used alone to generate accurate pressure values for all the different flow periods exhibited. Apart from the very early times, where the log approximation could be used, this paper shows that accurate pressure distribution beyond the early period can be computed using the Gauss-Legendre quadrature.
Results obtained using this method support the expected physical behaviour of oil and gas reservoirs and agree in principles with those published in literature.

INTRODUCTION

The most common and easily obtainable solution to the 3D diffusivity equations describing fluid flow in reservoirs containing horizontal wells is usually through the use of the Lord Kelvin's source functions and the Newman's product of the sources. This normally leads to an instantaneous pressure expression containing an integral to be evaluated with respect to time. The period of flow of interest determines the extent of integration to be performed and the method of solution to be applied. Application of the same integration for all periods yields results that may not make physical sense. For example, during the early transient flow period (infinite acting period) the best method is a suitable log approximation. Beyond the wellbore vicinity where the reservoir actual geometry and boundaries are beginning to be felt, other methods are employed to obtain the integral. The accuracy of results obtained, however, varies from one method to another and depends generally on the reservoir model.

In this paper the use of the Gauss-Legendre quadrature is discussed as an alternative to the existing methods. The mathematical development that follows describes the statement of the problem.

STATEMENT OF THE PROBLEM

The 3D diffusivity equation, for an anisotropic reservoir with a horizontal well, is given as follows for an unsteady oil flow :

$$k_x \frac{\partial^2 p}{\partial x^2} + k_y \frac{\partial^2 p}{\partial y^2} + k_z \frac{\partial^2 p}{\partial z^2} = \phi \mu c_t \frac{\partial p}{\partial t} \qquad (1)$$

If one dimension is considered from Eq. (1) as follows:

$$k_x \frac{\partial^2 p}{\partial x^2} = \phi \mu c_t \frac{\partial p}{\partial t} \quad (2)$$

the Laplace solution otherwise called the Green's function for an infinite linear reservoir is given as:

$$G(x, x', t) = \frac{1}{2\sqrt{\pi \eta_x t}} e^{-(x-x')^2 / 4\eta_x t} \quad (3)$$

where $\eta_x = {k_x}/{\phi \mu c_t}$. In a reservoir, the smallest domain is a well bore. Hence upon substitution of x_w for x′ in Eq. (3) where x is an arbitrary point in the reservoir and x_w is a point of reference (the wellbore) within the reservoir, we obtain the corresponding instantaneous source function for an infinite plane as:

$$s(x, t) = \frac{1}{2\sqrt{\pi \eta_x t}} e^{-\frac{(x-x_w)^2}{4\eta_x t}} \quad (4)$$

Eq.(4) is now a particular solution to Eq.(2). If the point of fluid withdrawal is at the point of symmetry in the reservoir and the reservoir is bounded such that there is no flow through the boundaries, the source is called a slab and its instantaneous source is obtained by integrating Eq. (3)from $x_w - x_f/2$ to $x_w + x_f/2$ to give

$$s(x, t) = \frac{1}{2}\left[erf\left(\frac{x + (x_w + \frac{x_f}{2})}{2\sqrt{\eta_x t}} \right) + erf\left(\frac{x - (x_w - \frac{x_f}{2})}{2\sqrt{\eta_x t}} \right) \right] \quad (5)$$

where x_f is the wellbore thickness. The equivalent sources can be written for the –y and –z directions. The general solution to Eq.(1) is therefore obtained by the Newman's product method as

$$s(x, y, z, t) = s(x, t) \,.\, s(y, t) \,.\, s(z, t) \quad (6)$$

If production or injection rate q is maintained in the wellbore then the pressure drop at any point in the reservoir is given as a continuous source as

$$\Delta p(x,y,z,\tau)=\frac{q}{\phi c_t}\int_0^t s(x,y,z,\tau)\,d\tau \qquad (7)$$

In terms of dimensionless parameters Eq. (7) may be written as

$$p_D(x_D,y_D,z_D,\tau)=2\pi h_D\int_0^{t_D} s(x_D,y_D,z_D,\tau)\,d\tau \qquad (8)$$

where

$$s(x_D, y_D, z_D, t_D) = s(x_D, t_D)\ .\ s(y_D, t_D)\ .\ s(z_D, t_D) \qquad (9)$$

$$s(x_D, t_D)=\frac{1}{2}\left[erf\left(\frac{1+x_D}{2\sqrt{\tau}}\right)+erf\left(\frac{1-x_D}{2\sqrt{\tau}}\right)\right] \qquad (10)$$

$$s(y_D, t_D)=\frac{1}{2\sqrt{\pi t_D}}\,e^{-(y_D-y_{wD})^2/4t_D} \qquad (11)$$

$$s(z_D, t_D)=\frac{1}{2\sqrt{\pi t_D}}\,e^{-(z_D-z_{wD})^2/4t_D} \qquad (12)$$

$$x_D=\frac{2x}{L}\sqrt{\frac{k}{k_x}} \qquad (13)$$

$$L_D=\frac{L}{2h}\sqrt{\frac{k}{k_x}} \qquad (14)$$

$$y_D=\frac{2y}{L}\sqrt{\frac{k}{k_y}} \qquad (15)$$

$$h_D=\frac{2h}{L}\sqrt{\frac{k}{k_z}} \qquad (16)$$

$$t_D=\frac{4kt}{\phi\mu c_t L^2} \qquad (17)$$

$k=\sqrt{k_x k_y}$ = horizontal permeability

These dimensionless parameters render **Eq.(1) to take the form:**

$$\frac{\partial^2 p_D}{\partial x_D^{\ 2}} + \frac{\partial^2 p_D}{\partial y_D^{\ 2}} + \frac{\partial^2 p_D}{\partial z_D^{\ 2}} = \frac{\partial p_D}{\partial t_D} \tag{18}$$

Eqs (10) to (12) are justified in the Appendix. Eqs. (9) to (17) are based on horizontal well half length, L/2. It is important to note that the wellbore storage and skin factors are not included in Eq.(8). Eq. (8) can be derived for various reservoir boundaries. No matter the reservoir being modelled the solution to the integral involved is common to all. The aim of this paper is to describe a numerical integration procedure, called the Gauss-Legendre quadrature, for performing integration of the type in Eq.(8), after identifying the relevant flow periods both in the wellbore and in the reservoir.

Ref. 3 suggested the use of Simpson rule for obtaining the integrals of the type in Eq. (8).The obvious problem associated with Simpson's rule for such integrals is the program overflow at very early times, especially at t_D=0.; pressures are infinite in magnitude. This is physically unacceptable! Refs. 4 and 5 following suggestions by Ref. 6 solved similar integrals as discrete areas under the curve of the integrand and time. This method gives fairly accurate results only for the early radial, and perhaps, part of the early linear periods. Results obtained for the early linear period gave some undesirable trends indicating that the reservoir model has an external recharge like a gas cap or an aquifer. This may not necessarily be the situation unless the reservoir model incorporates such boundaries. Refs. 7, 8 and 11 have used methods, though undisclosed, that seem to give a uniform slope 1.151 of pressure with time immediately after the nearest boundary had been felt. Although, this agrees mathematically with the simplified form of their dimensionless pressure expressions, the simplification may exclude certain important representations of the physical reservoir model, and are therefore apt to errors. Because all it involves is simply to identify the early period, and dimensionless pressure values beyond this period are simply obtained by a multiplication of the last value by a slope of 1.151. But results obtained from actual well test and utilized by Ref. 9 show that this trend is not accurate because several well test results and their plots do not show a similar uniform slope even beyond the early times. This constant slope is expected though, to depict pseudosteady flow, and there are more than one period of pseudosteady flow. The first encounter by flow transient of a boundary does not necessarily mean that all the wellbore and reservoir no-flow boundaries have been felt. As a matter of fact there are two major linear flow periods in a horizontal well and unless a reservoir is finite in dimensions a final and complete pseudosteady flow is not exhibited. This means that (1) for an infinite reservoir, a portion of the reservoir continues to exhibit infinite behavior even as other boundaries have manifested and hence (2) a hindered growth in p_D over time is not appropriate after only the nearest boundary has been felt. Appropriate flow periods should be identified and suitable expressions derived to represent them using the principle of superposition to account for all the events that occur in the wellbore since the first disturbance. Ref. 10 used the combined methods of Laplace and Fourier integral transforms successively to solve Eq. (18). Although good results were obtained, it was not without very tedious reservoir simulation studies. Obtaining the inverse of all the transforms was obviously impossible without some serious simplifying assumptions. The method proposed here is not tedious mathematically and does not involve building a simulator to understand the relationships between parameters. The integral is evaluated in a straightforward manner.

APPLICATION OF THE GAUSS-LEGENDRE QUADRATURE

We consider fluid flow through a bounded anisotropic reservoir. The solution is given for all periods of flow by Ref. 11 as:

$$P_D(x_D, y_D, z_D, z_{wD}\ L_D, t_D) = \sqrt{\frac{\pi}{4}}\sqrt{\frac{k}{k_y}}\int_0^{t_D}\left[erf\left(\frac{1+x_D}{2\sqrt{\tau}}\right)+erf\left(\frac{1-x_D}{2\sqrt{\tau}}\right)\right]\left[e^{-y_D{}^2/4\tau}\right]\times$$
$$\left[1+2\sum_{n=1}^{\infty}\exp\left(-l^2\pi^2 L_D\tau\right)\cos l\pi z_{wD}\cos l\pi z_D\right]\frac{d\tau}{\sqrt{\tau}} \qquad (19)$$

Following Ref. 2, the dimensionless pressures in Eq.(19) is now written as follows:

$$p_D(x_D, y_D, z_D, t_D) = -\frac{\beta}{4L_D}Ei\left[-\left(\frac{(z_D-z_{wD})^2/L_D{}^2+y^2{}_{wD}}{4t_D}\right)\right]$$
$$+\frac{(t_D-t_{De})}{2}\sqrt{\frac{\pi k}{4k_y}}\sum_{i=0}^{n}w_i f\left[erf\left(\frac{(1+x_D)\sqrt{2}}{2\sqrt{z_i t_D+t_D+t_{De}}}\right)+erf\left(\frac{(1-x_D)\sqrt{2}}{2\sqrt{z_i t_D+t_D+t_{De}}}\right)\right]\times$$
$$e^{-\frac{y^2{}_D}{2(z_i t_D+t_D+t_{De})}}\times\left[1+2\sum_{n=1}^{\infty}\exp(-l^2\pi^2 L_D(z_i t_D+t_D+t_{De})\cos l\pi z_D\cos l\pi z_{wD}\right]\times$$
$$\frac{dz_i\sqrt{2}}{\sqrt{z_i t_D+t_D+t_{De}}} \qquad \textbf{(20)}$$

The second term on the RHS is the actual Gauss-Legendre quadrature. z_l are the roots and w_l are the weight factors of the quadrature. n is the number of points chosen for computation. The choice of n depends on the degree of the polynomial function, $f(p_{wD})$. z_l and w_l are listed in Ref. 2. The flow chart for the computing p_D is shown in Fig. 1.

The wellbore parameters used in the example are as follows: L_D= 0.25, $r_{wD} = 10^{-4}$, z_{wD} = 0.5, x_D = 0.732 (infinite conductivity horizontal well). The dimensionless wellbore radius r_{wD} is the same in all directions and it is the same as the wellbore position along the y-axis. Hence $y_{wD} = r_{wD} = 10^{-4}$. Substituting these data into Eq. (20) gives the dimensionless wellbore pressure, p_{wD} for several values of t_D. β=1.0.

Suggestions by Ref. 3 on the computation of the summation did not make much difference in the overall computed p_{wD} with only a few terms considered in the summation.

RESULTS AND DISCUSSION

The argument of the exponential integral Ei, in Eq. (20) shows that the z-direction is also undergoing infinite behavior during this period $t_D \leq t_{De}$. The period of validity of the early time period in Eq.(20) depends strongly upon the magnitudes of x_D, z_D, z_{wD}, L_D and the reservoir anisotropy and it is given as

$$t_{De} \leq \min \begin{cases} \delta^2{}_D / 20 \\ (z_D + z_{wD})^2 / (20L^2{}_D) \\ (z_D + z_{wD^{-2}})^2 / (20L^2{}_D) \end{cases} \quad (21)$$

where $\delta = 1 - x_D$ for isotropic reservoir under consideration. Thus for data given above, $t_De \leq 3x10^{-3}$. That is , the exponential integral in Eq. (20) can be approximated as:

$$p_D(x_D\ y_D, z_D, t_D) = -\ln(1.781x) \quad (22)$$

for an isotropic reservoir where

$$x = \left(\frac{(z_D - z_{wD})^2 / L^2{}_D + y_D{}^2}{4t_D} \right) \quad (23)$$

provided $t_D \leq t_{De}$. Results obtained are tabulated in Table 1 along with those of Ref.11.

Expectedly, results are same during the infinite acting flow period. Results during this time are the Ei functions of the argument given by Eq.(3). Ref.11 shows results that were fixed using a slope of 1.151 immediately after the early time; that is, during the onset of the first pseudosteady state occasioned by the passage of pressure transient across the nearest wellbore boundary. It therefore means that even as other wellbore boundaries are being felt it does not matter the amount of wellbore volume that now contributes to flow. This is not an acceptable representation of the wellbore. At the onset of the first pseudosteady flow a unique and uniform change in pressure with time is established. This some what stabilized flow is disturbed when other new well boundaries are felt especially for horizontal wells. This introduces a new regime of pressure gradient in the wellbore pressure. The change in pressure gradient is expectedly more pronounced than in the first instance due to a larger area of the wellbore now contributing to flow. It is therefore not appropriate to impose a fixed slope after the disappearance of the infinite period. This is the exact state of affair represented by the results obtained using the Gauss-Legedre quadr... as si Table 1. To be factual, there cannot be a reasonably prolonged pseudo... behaviour until the farthest wellbore boundary is felt. Furthermore, a perma... psuedosteady state is achieved if flow lasts enough for the farthest reservoir boundary to be felt. This may occur for infinite time if the reservoir is modeled as infinite.

t_D	P_{WD} from Ref.11	P_{WD} using Gauss-Legendre quadrature
10^{-5}	7.717	7.72
10^{-4}	10.02	10.02
10^{-3}	12.32	12.32
10^{-2}	14.61	14.29
10^{-1}	16.58	17.50
1	17.99	20.18
10	19.07	22.91
10^{2}	20.21	25.32
10^{3}	21.37	27.64
10^{4}	22.52	29.94
10^{5}	23.67	32.24

Table 1.:Comparison of Dimensionless Wellbore Pressure Data for Ref.11 and the Gauss-Legendre quadrature

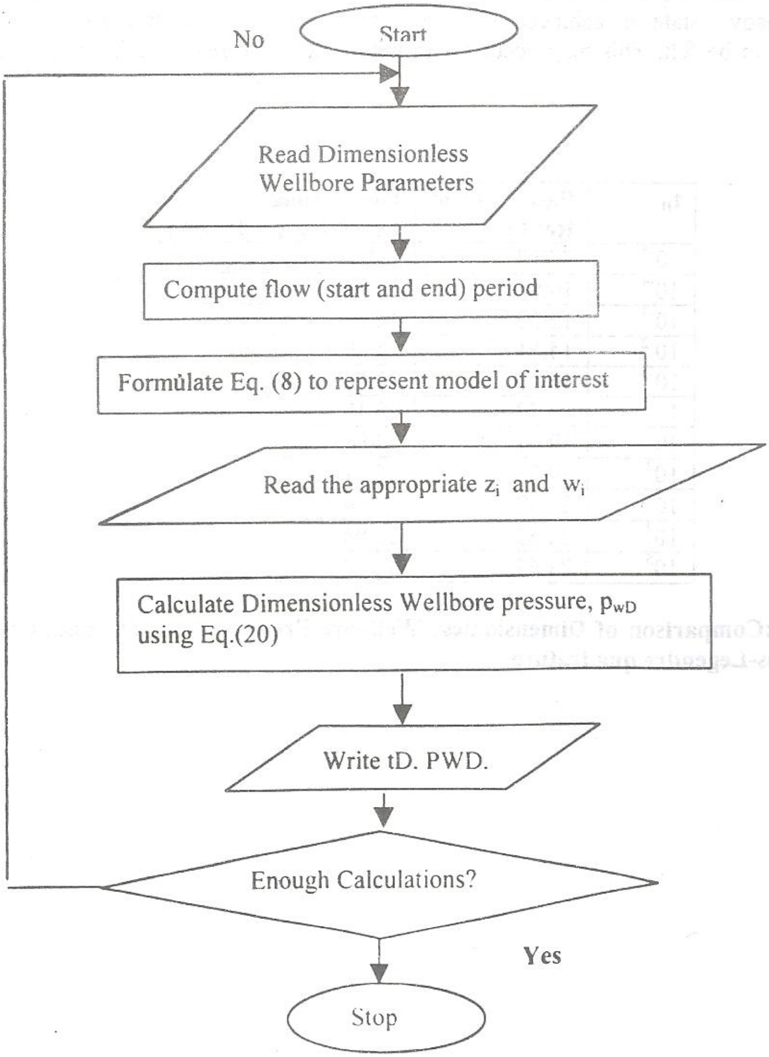

Fig.1.: Flow Chart for computing p_{wD}

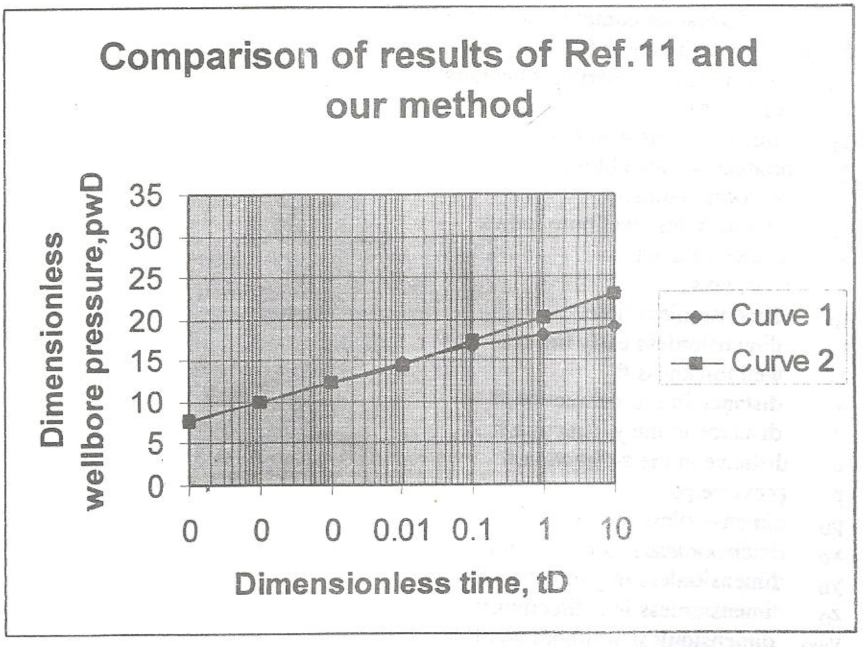

Fig. 2.: Comparison of results of Ref.11 and those obtained using the Gauss-Legendre quadrature
Curve 1 from Ref. 11; Curve 2 from Gauss-Legendre

CONCLUSION

For the first time a comprehensive and straightforward method has been presented to evaluate integrals associated with horizontal well flow problems. This has eliminated the rigorous mathematical labour which often leads to inaccurate solutions when solving related integrals. In accepting this method, however, full account must be made for the error associated with the quadrature. If these errors are accounted for, more accurate results could be obtained to describe the pressure distribution in a horizontal wellbore with time for any given set of wellbore and reservoir parameters using the Gauss-Legendre quadrature.

Nomenclature

c_t total formation compressibility,psi^{-1}
h reservoir thickness,ft
h_D dimensionless reservoir thickness
L well length,ft
L_D dimensionless well length
q production rate,bbl/day
r_w wellbore radius,ft
r_{wD} dimensionless wellbore radius
s source function
t time,days
t_D dimensionless time
t_{De} dimensionless early time
x_f well thickness,ft
x distance in the x-direction,ft
y distance in the y-direction,ft
z distance in the z-direction,ft
p pressure,psi
p_D dimensionless pressure
x_D dmensionless in x direction,ft
y_D dimensionless in y direction,ft
z_D dimensionless in z direction,ft
y_{wD} dimensionless wellbore width,ft
k total horizontal permeability,md
k_x permeability in the x-direction,md
k_y permeability in the y-direction,md
k_z permeability in the z-direction,md
Δp pressure drop,psi
τ dummy integration variable
μ oil viscosity,cp
ϕ reservoir porosity,fraction
η diffusivity constant, mdpsi/cp

APPENDIX

Equations (11) and (12) in the text are not having the same form as Equation (10) considering the argument below:
Green's functions are general Laplace solutions for arbitrary positions in the reservoir. Source functions on the other hand are particular solutions as stated in the paper. Source functions as defined by Eqs. (11) and (12), depend upon one space variable only and do not, in general, satisfy all properties of the Green's functions. The sources in y and z- axes of the reservoir model are infinite sources in an infinite reservoir. Hence their respective source functions are simply written by **substituting the arbitrary position in their Green's functions with a wellbore position,**

without integration (Ref. 1). The source function from **the x-axis is a slab** source of thickness $x_l/2$ or L/2 in an infinite reservoir. Hence, it is **obtained by integrating the corresponding Green's function** from(x_w-x_f/2) to **(x_w+x_f/2) (see Ref. 1).** X_w is the plane of symmetry generally regarded as the point in **the wellbore** where fluid flow takes place. The integration is as follows

$$s(x,t) = \int_{x_w - x_f/2}^{x_w + x_f/2} G(x, x't)dx' = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x + (x_w + \frac{x_f}{2})}{2\sqrt{\eta_x t}} \right) + \operatorname{erf}\left(\frac{x - (x_w - \frac{x_f}{2})}{2\sqrt{\eta_x t}} \right) \right]$$

where

$$\eta_x = k\sqrt{\mu\phi c_l}$$

With the substitution of the defined dimensionless parameter and noting **that there** can not be a radius along the well length $x_{WD} = 0$ and $k = k_x$ for isotropic **reservoir** and hence

$$s(x_D, t_D) = \frac{1}{2}\left[erf\left(\frac{1 + x_D}{2\sqrt{t_D}} \right) + erf\left(\frac{1 - x_D}{2\sqrt{t_D}} \right) \right]$$

Note finally that τ and t_D both connote dimensionless time **parameters.**

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