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ANOTHER NUMERICAL METHOD FOR LINEAR INTEGRAL EQUATION

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ABSTRACT

Another Method for numerical solution of a linear integral equation when the range of integration is finite is described. It consists essentially of using Clanshaw-Curtis rule which is predictable and saves a lot of time when the need of increase in quadrature point arises. Its application is demonstrated.

INTRODUCTION

In this paper the linear integral equation which is generally represented as

$$\phi(x) = f(x) + \int_a^b k(x,t)\phi(t)dt \qquad (1)$$

is considered. Since the definite integral can be closely approximated by any one of the several quadratic formulas J. Scanborough in [1] stated that (1) may be written in the form

$$\phi(x) = f(x) + (b-a)\sum_i C_i k(x,t_i)\phi(t_i) \qquad (2)$$

where $t_1, t_2, \ldots t_n$ are subsitituting division points of the interval (a,b) and these C's are weighing coefficients whose values depend on the type of quadrature formulas used. Since (2) must hold for all values of x in the interval (a, b) it must hold for $x = t_1$, $x = t_2 \ldots x = t_n$.
Hence from (2) we get n equations of the type

$$\phi(t_i) = f(t_i) + (b-a)[C_1 k(t_i,t_1)\phi(t_1) + C_2 k(t_i,t_2)\phi(t_2) + \ldots C_n k(t_i,t_n)\phi(t_n), \quad i = 1,2,\ldots n \qquad (3)$$

Letting $\phi(t_1) = \phi_i$ and $f(t_i) = f_l$ for brevity then the system (3) becomes

$$\left.\begin{aligned}\phi_1 &= f_1 + (b-a)\sum_{i=1}^{n} C_1 k(t_i, t_1)\phi_1 \\ \phi_2 &= f_2 + (b-a)\sum_{i=1}^{n} C_1 k(t_2, t_1)\phi_i \\ \phi_n &= f_n + (b-a)\sum_{i=1}^{n} C_1 k(t_n, t_i)\phi_i\end{aligned}\right\} \qquad (4)$$

Equation (4) are system of n linear equations is n unknowns $\phi_1, \phi_2 \ldots \phi_n$ which can be solved. Unlike in quadrature formula where the functional values $\phi_1, \phi_2 \ldots \phi_n$ are given at point of subsititutingdivision in integral equation the values must be found out as part of process of solving the equation and the equations for finding then are obtained by putting $x = t_1, x = t_2, \ldots x = t_n$.

The above numerical process was demonstrated by Scarborough in [1] in the following examples. (1) Solve the integral equation

$$u(x) = \frac{5x}{6} - \frac{1}{9} + \frac{1}{3}\int_b^a (t+x)u(t)dt$$

Applying Simpson's rule for n = 3 or h = ½ he got

$$u(x) = \frac{5x}{6} - \frac{1}{19} + \frac{1}{3}\frac{1}{3}\frac{1}{2}\left[(t_1 + x)u(t_1 + 4(t_2 + x))u(t_2) + (t_3 + x)u(t_3)\right] \qquad (5)$$

since (5) must hold for all values of x from 0 to 1 it holds for $x = t_1$, $x = t_2$, $x = t_3$. Hence from 5 we get

$$\begin{aligned} u(t_1) &= \frac{5t_1}{6} - \frac{1}{19} + \frac{1}{18}\left[2t_1 u(t_1) + 4(t_2 + t_1)u(t_2) + (t_2 + t_1)u(t_2)\right] \\ u(t_2) &= \frac{5t_2}{6} - \frac{1}{19} + \frac{1}{18}\left[(t_1 + t_2) + 4(2t_2)u(t_2) + (t_3 + t_2)u(t_3)\right] \qquad (6) \\ u(t_3) &= \frac{5t_3}{6} - \frac{1}{19} + \frac{1}{18}\left[(t_1 + t_3) + u(t_2)4(t_2 + t_3)u(t_2) + 2 + u(t_3)\right] \end{aligned}$$

putting $t = 0$, $t_2 = ½$, $t_3 = 1$ and writing $u(t_1) = u_1$ we get

$$\left.\begin{aligned} u_1 &= -\frac{1}{9} + \frac{1}{18}[2u_2 + u_3] \\ u_2 &= -\frac{3}{12} + \frac{1}{18}\left[\frac{u_1}{2} + 4u_2 + \frac{3}{2}u_3\right] \\ u_3 &= -\frac{1}{6} + \frac{1}{18}[u_1 + 6u_2 + 2u_3] \end{aligned}\right\} \quad (7)$$

Simplifying (7) we have

$$\left.\begin{aligned} 36u_1 - 4u_2 - 2u_3 &= -u \\ -u_1 + 28u_2 - 3u_3 &= 11 \\ -2u_1 - 12u_2 + 32u_3 &= 26 \end{aligned}\right\} \quad (8)$$

and solving gives $u_1 = 0$, $u_2 = ½$ and $u_3 = 1$, and substituting these values in (5) gives

$$u(x) = \frac{5x}{6} - \frac{1}{9} + \frac{1}{18}\left[0 + 4\left(\frac{1}{2} + x\right)\frac{1}{2} + (1 + x)(1)\right] = x \quad (9)$$

i.e. $u(x) = x$

Example 2

Solve the integral equation

$$u(x) = 2x + \frac{1}{5}\int_{-2}^{3}(x - t)u(t)\,dt \quad (10)$$

Using the three point Gauss rule as in [2] and [3] we have the following on transformation from the interval (-2, 3) to (-½, ½).

We have $t = ½ + 5v$, and

$$x = ½ + 5w \quad (11)$$

substituting (11) in (10) we have

$$u\left(\frac{1}{2} + 5w\right) = 1 + 10w + \int_{-½}^{½}(w - v)u\left[\left(\frac{1}{2} + 5v\right)5dv\right] \quad (12)$$

on putting $u\left(\frac{1}{2} + 5w\right) = \phi(w)$ and $u\left(\frac{1}{2} + 5v\right) = d(v)$

then (12) becomes

$$\phi(w) = 1 + 10w + 5\int_{-1/2}^{1/2} (w - v)\phi(v)\, dv \tag{13}$$

replacing the integral with Gauss formula we have

$$\phi(w) = 1 + 10w + 5[R(w - v_1)\phi(v_1) + R_2(w - v_2) + R_3(w - v_3)\phi(v_3)] \tag{14}$$

since (14) must hold for all values of w from – ½ to ½ it must hold for $w = v_1$, $w = v_2$ and $w = v_3$. Hence on substituting in (14) we get the equation.

$$\begin{aligned} \phi(v_1) &= 1 + 10v_1 + 5[R_2(v_1 - v_2)\phi(v_2) + R_3(v_1 - v_3)\phi(v_3)] \\ \phi(v_2) &= 1 + 10v_2 + 5[R_1(v_2 - v_2)\phi(v_1) + R_2(v_2 - v_3)\phi(v_3)] \\ \phi(v_3) &= 1 + 10v_3 + 5[R_1(v_3 - v_1)\phi(v_1) + R_2(v_3 - v_2)] \end{aligned} \tag{15}$$

For the Gauss 3 point formula $v_1 = -\frac{1}{2}\sqrt{\frac{3}{5}}, v_2 = 0, \quad v_3 = \sqrt{\frac{3}{5}}$

$$R_1 = \frac{5}{18},\ R_2 = \frac{4}{9},\ R_3 = \frac{5}{18}$$

substituting in (15) $\phi(v_1) = \phi_1 \ \phi(v_2) = \phi(v_3) = \phi_3$ and solving (15) we get

$$\begin{aligned} \phi_1 &= -\frac{2}{37}\left(19 + 9\sqrt{5}\right) \\ \phi_2 &= -\frac{38}{37} \\ \phi_3 &= -\frac{2}{37}\left(19 - 9\sqrt{15}\right) \end{aligned} \tag{16}$$

substituting in (14) these values of ϕ_1, ϕ_2 and ϕ_3 R's and V's we get

$$\phi_1(w) = \frac{180}{37} - \frac{38}{37} \tag{17}$$

and since $w = \frac{x - \frac{1}{2}}{5}$ the final solution of (10) is

$$u(x)_1 = \frac{180}{37}\frac{\left(x-\frac{1}{2}\right)}{5} - \frac{38}{37} = \frac{36x}{37} - \frac{56}{37} = \frac{4}{37}(9x-14) \qquad (18)$$

FORMULATION OF THE NEW METHOD USING CLENSHAW-CURTIS RULE

Clenshaw Curtis formula in [4] states that a given function f(x) can be expanded in terms of Chebyshev polynomial such that

$$f(x) \equiv F(t) = \frac{1}{2}a_o + a_1T_1(t) + a_2T_2(t) \ldots \frac{1}{2}a_NT_N(t) \qquad (19)$$

$$a \le x \le b \ where T_n = \cos(r\cos^{-1}t) \ and \ t = \frac{2x-(b+a)}{b-a}$$

with the setting above Clenshaw Curtis rule states that

$$\int_{-1}^{1} F(t)dt = 2(b_1 + b_3 + b_5 \ldots) \qquad (20)$$

where
$$b_r = \frac{a_{r-1} - a_{r+1}}{2r} \qquad (21)$$

$$a_r = \frac{2}{N}\sum_{s=o}^{N}{}'' F_3 \cos\frac{\pi rs}{N} \qquad (22)$$

$$F_s = F\left(\cos\frac{\pi rs}{N}\right)$$

and Σ" denotes a finite sum whose first and last terms are to be halved. So using a 2 point Clenshaw Curtis rule with b_5 being negligible we have

$$\int_{-1}^{1} f(t)dt = 2(b_1 + b_3) \qquad (23)$$

using (21) in (23) we get

$$\int_{-1}^{1} F(t) = \left(\frac{3a_o - a_2 - a_4}{3}\right) \qquad (24)$$

for s = 1, 2, 3, 4, $\cos\left(\frac{\pi s}{4}\right) = 1$ for s = 0

$$\cos\left(\frac{\pi s}{4}\right) \qquad = .70710681, \qquad s = 1$$

$$= 0 \qquad s = 2$$

$$= -.70710681 \qquad s = 3$$

$$= -1 \qquad s = 4$$

using (22) in 24) we get

$$\int_{-1}^{1} F(t)dt = \frac{F_0}{6} + \frac{7}{6}F_1 + F_2 + \frac{F_3}{6} + \frac{5}{6}F_4$$

APPLICATION OF THE NEW METHOD

Applying (25) to equation (10) with the transformation

$$t = \frac{2V - (b+a)}{b-a} \text{ and } x = \frac{2w - (b+a)}{b-a}$$

We have

$$\phi(w) = sw + 1 + \frac{5}{4}\left\{(w - v_o)\frac{F_o}{6} + \frac{7}{6}(w - v_1)f_1 + (w - v_2)f_2 + \frac{7}{6}(w - v_3)f_3 + \frac{5}{6}(w - v_4)f_{41}\right\} \tag{26}$$

putting

$w = v_0$, $w = v_1$, $w = v_2$, $w = v_3$ and $w = v_4$

and

$f_o = \phi(v_o)$, $F_1 = \phi(v_1)$, $F_2 = \phi(v_2)$, $F_3 = \phi(v_3)$, $F_4 = \phi(v_4)$ and $\phi(v_i) = \phi_i$ in (26)

we get

$$\left.\begin{aligned}
\phi_o &= 5v_o + 1 + \frac{5}{4}\left\{\frac{7}{6}(v_o - v_1)\phi_1 + (v_o - v_2)\phi_2 + \frac{7}{6}(v_o - v_3)\phi_3 + \frac{5}{6}(v_o - v_4)\phi_4\right\} \\
\phi_1 &= 5v_1 + 1 + \frac{5}{4}\left\{\frac{1}{6}(v_1 - v_o)\phi_o + (v_1 - v_2)\phi_2 + \frac{7}{6}(v_1 - v_3)\phi_3 + \frac{5}{6}(v_1 - v_4)\phi_4\right\} \\
\phi_2 &= 5v_2 + 1 + \frac{5}{4}\left\{\frac{1}{6}(v_2 - v_o)\phi_o + \frac{7}{6}(v_2 - v_1)\phi_1 + \frac{7}{6}(v_2 - v_3)\phi_3 + \frac{5}{6}(v_2 - v_4)\phi_4\right\} \\
\phi_3 &= 5v_3 + 1 + \frac{5}{4}\left\{\frac{1}{6}(v_3 - v_o)\phi_o + \frac{7}{6}(v_3 - v_1)\phi_1 + \frac{7}{6}(v_3 - v_3)\phi_2 + \frac{5}{6}(v_3 - v_4)\phi_4\right\} \\
\phi_4 &= 5v_4 + 1 + \frac{5}{4}\left\{\frac{1}{6}(v_4 - v_o)\phi_o + \frac{7}{6}(v_4 - v_1)\phi_1(v_4 - v_2)_2\phi + \frac{7}{6}(v_4 - v_3)\phi_7\right\}
\end{aligned}\right\} \tag{27}$$

substituting $v_o = 1$, $v_1 = .70710681$, $v_2 = 0$, $v_3 = -.707106781$ and $v_4 = -1$ into (17) and solving for ϕ_o, ϕ_1, ϕ_2, ϕ_3, and ϕ_4 we again get $U(x) = \frac{4}{37}(9x - 14)$ which is the solution of (10).

CONCLUSION

The attraction for applying Clenshaw-Curtis rule is that the a's are generated automatically and an increase in the use of quadrature point saves time as only the additional points may be generated unlike the Gauss rule where an increase in the number of quadrature point requires an entirely new set of points. Moreover whether the system of equations (27) is large or small (which depends on the number of quadrature points) the Clenshaw-Curtis rule which is more consistent gives more accurate result.

REFERENCE

[1] J. B. Scarborough. Numerical Analysis, The John's Hoplans Press, (1966).
[2] F. B. Hilderbrand. Introduction to Numerical Analysis, New York McGraw-Hill (1956).
[3] Z. Kopal. Numerical Analysis, London Chap Mall and Hall (1955).
[4] C. W. Clenshaw and A. R. Curtis. A method of Numerical Integration, Numerishe Mathematik 2, 197 – 205 (1960).