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# EXACT (PERMUTATION) DISTRIBUTION FOR THE $W_N$ STATISTIC

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## ABSTRACT

A test statistic called the $W_N$ statistic as introduced by Ogbonmwan (1983), which is based on functions of ranks, forms the basis of this work. The test statistic involves a Multi-Sample Testing Procedure (M-STP) that takes into consideration all the observations of the combined $p \geq 2$ samples (treatments) in a one-way analysis of variance (ANOVA) problem. The work here provides the exact (permutation) distribution for the $W_N$ statistic.

## 1. INTRODUCTION

Let $n_i$, $i = 1(1)p$ be the number of observations in the ith sample (treatment) of a one-way ANOVA problem.

Let $x_{i1}, x_{i2}, \ldots, x_{in_i}$ ; $i = 1(1)p$ be the ordered observations of the ith sample from a population with continuous cumulative distribution function (cdf) $F_i(x)$, $i = 1(1)p$.

$$\text{Let } \lambda_{Ni} = {}^{n_i}\!/_{N} \quad (= n / N) \tag{1.1}$$

$$\text{where } N = \sum_{i=1}^{p} n_i \tag{1.2}$$

$$\text{Let } F(x) = 1 / n_i \,(\,\#\text{ of } x_{ij} \leq x,\ j = 1\,2, \ldots, n_i) \tag{1.3}$$

where # stands for "number", be the sample cdf of the $n_i$ observations in the ith sample. Therefore, the combined sample cdf is:

$$H(x) = \lambda_{N1}F_1(x)+\lambda_{N2}F_2(x)+\cdots+\lambda_{NP}F_p(x)=\sum\lambda_{Ni}F_i(x)$$

For the formulation of the class of rank order statistic, which is used in the development of $W_N$ statistic, we let

$$X_i = (x_{i1}, x_{i2}, \ldots, x_{in_i}),\ i = 1(1)p \tag{1.4}$$

$$E_N = (E_{N1}, E_{N2}, \ldots, E_{N,P}) \tag{1.5}$$

Where

$$E_{N,\alpha} = J_N\left(\frac{\alpha}{N+1}\right), \; 1 \le \alpha \le N \tag{1.6}$$

and $J_N$ is defined as in Ogbonmwan (1983), either as:

$$J_N(u) = u \qquad (\text{Wilcoxon version})$$

or $$J_N(u) = \Phi^{-1}(u) \quad (\text{Van De Warden version})$$

Let us write

$$\bar{E}_N = \frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha} \tag{1.7}$$

and $$A_N^2 = \frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha}^2 - \bar{E}_N^2 \tag{1.8}$$

Let $Z_{N,\alpha}^{(i)} = 1$, if the $\alpha$ th smallest observation in the combined ranking of all the N observations (in the data formation) is from the ith sample and

Let $Z_{N,\alpha}^{(i)} = 0$ otherwise for $\alpha = 1(1)N$.

We now consider (for the ith sample) the following class of Chernoff-Savage type of rank order statistics:

$$h_N(X_i) = \frac{1}{n_i}\sum_{\alpha=1}^{N} E_{N,\alpha}\, Z_{N,\alpha}^{(i)} \; ; \; i = 1(1)p \tag{1.9}$$

It is convenient to consider the Wilcoxon version of the definition of $J_N$ which is given as $J_N(u) = u$. Thus:

$E_{N,\alpha} = \dfrac{\alpha}{N+1}$ and hence (1.9) simplifies to

$$h_N(X_i) = \frac{1}{(N+1)n_i}\sum_{\alpha=1}^{N} \alpha\, Z_{N,\alpha}^{(i)} \; ; \; i = 1(1)p \tag{1.10}$$

From Ogbonmwan (1983),

$$\{(nh_N(X) - \bar{E}_N) / A\} \xrightarrow{L} N(0,1)$$

where $\bar{E}_N$ and $A$ are normalising constants.

Now, for i = 1(1)p samples, the $W_N$ statistic is given as:

$$W_N = \max_{1 \le i, j \le P} \left[ n^{1/2} A_N^{-1} \left| h_N(X_i) - h_N(X_j) \right| \right] \quad (1.11)$$

or

$$W_N = n^{1/2} A_N^{-1} \left[ \max h_N(X_i) - \min h_N(X_i) \right], \; i = 1(1)p \quad (1.12)$$

Observe that $h_N(X_i)$ is the function of ranks of a particular sample in any configuration. This presupposes that all the distinct configurations must be obtained first. This involves the permutation of the elements of a matrix.

In Ogbonmwan (1983), the asymptotic distribution for the $W_N$ statistic was given via the following theorem:

**Theorem 1.1**

Assume that Equations (1.1) – (1.12) are satisfied, then under the null hypothesis

$H_0$: $F_1(x) = F_2(x) = \ldots = F_k(x)$ for all real x

$$\lim_{N \to \infty} \text{Prob}\{W_N \le t\} = \chi_p(t) \quad (1.13)$$

where $\chi_p(t)$ is the cdf of the sample range in a sample of size p drawn from a standardized normal distribution.

There has been no theoretical proof for the exact distribution of the $W_N$ statistic. However, there exist a permutation approach for the exact distribution of the $W_N$ statistic. But, due to the difficulty in generating all the configurations needed in the permutation approach, the discussions on it have ended in the realm of "speculation".

The focus of this work was inspired by the use of a computer-aided algorithm to generate the exact (permutation) distribution for the $W_N$ statistic. This algorithm has made us to realise the unthinkable and the needed desire to go beyond the realm of speculation.

## 2. A PERMUTATION METHOD FOR THE $W_N$ STATISTIC

A permutation method of evaluating the exact distribution of $W_N$ is as follows:

Let $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$ i = 1(1)p

and let us write

$$X_N = (X_1, X_2, X_3, \ldots, X_P)$$

i.e. $$X_N = \begin{pmatrix} x_{11} & x_{12} & \cdots\cdots\cdots & x_p \\ x_{21} & x_{22} & \cdots\cdots\cdots & x_{2p} \\ \vdots & \vdots & \vdots \;\; \vdots & \vdots \\ \vdots & \vdots & \vdots \;\; \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots\cdots\cdots & x_{np} \end{pmatrix}$$

Under the null hypothesis, $X_N$ is composed of N independent and identically distributed random variables and hence conditioned on any given $X_N$, all the possible (N!) permutations of the variates among themselves are equally likely.

Therefore, we have,

$$\frac{N!}{\prod_{i=1}^{p}(n_i)!}$$

possible partitionings (configurations) of those N variables of p subsets of size $n_i$, i = 1(1)p which are equally likely, each having the conditional probability,

$$\left(\frac{N!}{\prod_{i=1}^{p}(n_i)!}\right)^{-1}$$

Consider the set of all these partitionings and for each one of them, compute the value of $W_N$ and hence consider the probability of the different values of $W_N$. Thus we have the table

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $W_N$ | | | | | | | | | | |
| $P(W_N)$ | | | | | | | | | | |

A point to worry about in the realisation of this exact (permutation) distribution of $W_N$ is that the amount of computations needed for the implementation is considerably enormous. For example, a data set consisting of just four treatments with four observations per treatment demands as much as $\frac{16!}{4!4!4!4!} = 63{,}063{,}000$ partitionings for a complete enumeration.

Considering this associated complexity in partitionings for a complete enumeration, we now offer a mathematical algorithm that will ensure that a complete and systematic enumeration of the partitionings is carried out.

## 2.1 Mathematical Formulation of the Algorithm: (Balanced cases, where $n_1 = n_2 = n_3 = n_p = n$)

***Two Treatments***

For all possible partitionings of the N variables of p subsets of size $n_i$ $i = 1(1)p$, we have to systematically develop a pattern necessary for the algorithm.

Let us examine a matrix of two variables. We are expected to have

$$\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i} \quad \text{partitionings}$$

where n is the number of variates in each treatment (column) if we assume that the two samples have equal number of variates.

For unequal samples, the number of partitionings is:

$$\sum_{i=0}^{\min(n_1,n_2)}\binom{n_1}{i}\binom{n_2}{i}$$

It follows therefore that for any matrix with two columns, $\sum_{i=0}^{\min(n_1,n_2)}\binom{n_1}{i}\binom{n_2}{i}$, can be used in enumerating all the configurations, regardless of whether the samples are of equal or unequal sizes.

***Three Treatments***

First permute the elements of any two of the treatments, and then find the number of ways we can permute any $n_3$ elements of the 3 treatments.

Expressing the above mathematically, a complete enumeration yields,

$$\binom{3n}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{i}$$

where n is the number of elements in any treatme

***Four Treatments***

By following the same procedure as done for the case of three treatments, we have that a complete enumeration of all the configurations for four treatments is achieved by using

$$\binom{4n}{n}\binom{3n}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{i}$$

Continuing in this manner, for p ≥ 3 treatments, we have that the number of configurations is

$$\prod_{j=3}^{p}\binom{jn}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{i}=\prod_{j=3}^{p}\binom{jn}{n}\sum_{i=0}^{n}\binom{n}{i}^2$$

But, $\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$ [see Feller (1968)].

Therefore, for p ≥ 3 treatments, number of configurations is

$$\prod_{j=3}^{p}\binom{jn}{n}\binom{2n}{n} = \prod_{j=1}^{p}\binom{jn}{n}$$

**Theorem 2.1**

**The number** of partitionings of the elements of p samples of equal sizes required **for the distribution** of the $W_N$ statistic is given by $\prod_{j=1}^{p}\binom{jn}{n}$, where n is the number **of elements in** each sample.

**Proof (By Induction)**

We simply need to show that $\prod_{j=1}^{p}\binom{jn}{n} = \frac{N!}{\prod_{i=1}^{k}(n_i!)}$ = number of possible partitionings of N variates of p subsets, each of size $n_i$, which are equally likely. Consider the balanced case for which $n_1 = n_2 = \ldots = n_k = n$.

Take $p = 1$: $\prod_{j=1}^{1}\binom{jn}{n} = \binom{n}{n} = \frac{n!}{n!} = \frac{N!}{\prod_{j=1}^{1}(n!)}$, where N = n for p = 1

$$\binom{n_1+n_2+n_3}{n_3}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i}=\binom{\sum_{r=1}^{3}n_r}{n_3}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i} \tag{2.2}$$

which gives the number of configurations for a three sample case.

***Four Treatments***

Adopting (2.2) to cater for four treatments, we have

$$\binom{n_1+n_2+n_3+n_4}{n_4}\binom{\sum_{r=1}^{3}n_r}{n_3}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i}=\binom{\sum_{r=1}^{4}n_r}{n_4}\binom{\sum_{r=1}^{3}n_r}{n_3}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i}$$

$$=\prod_{j=3}^{4}\binom{\sum_{r=1}^{j}n_r}{n_j}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i} \tag{2.3}$$

(2.3) is the number of configurations for a four sample case.

Continuing in this manner, for any p samples, of unequal sizes, we have

$$\text{Number of partitionings} = \prod_{j=3}^{p}\binom{\sum_{r=1}^{j}n_r}{n_j}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i} \tag{2.4}$$

But $\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i}=\binom{n_1+n_2}{n_2}=\binom{\sum_{r=1}^{2}n_r}{n_2}$. Proof is by induction.

Substituting, we have (2.4) simplifying to

$$\prod_{l=3}^{p}\binom{\sum_{r=1}^{l}n_r}{n_l}\binom{\sum_{r=1}^{2}n_r}{n_2}=\prod_{l=2}^{p}\binom{\sum_{r=1}^{l}n_r}{n_l}=\prod_{l=1}^{p}\binom{\sum_{r=1}^{l}n_r}{n_l} \tag{2.5}$$

Observe that if we allow $n = n_1 = n_2 = n_3 = \ldots = n_k$ in (2.5) we have, $\prod_{l=1}^{p}\binom{jn}{n}$

which agrees with the expression derived earlier for the balanced case.

$$p = 2: \quad \prod_{j=1}^{2}\binom{jn}{n} = \binom{n}{n}\binom{2n}{n} = \frac{(2n)!}{n!n!} = \frac{N!}{\prod_{j=1}^{2}(n!)}, \text{ where } N = 2n$$

Assume true for $p = k$

$$\text{i.e.} \quad \prod_{j=1}^{k}\binom{jn}{n} = \frac{nk!}{\prod_{i=1}^{k}(n!)} = \frac{N!}{\prod_{i=1}^{k}(n!)}, \text{where } N = nk$$

For $p = k + 1$,

$$\prod_{j=1}^{k+1}\binom{jn}{n} = \left[\prod_{j=1}^{k}\binom{jn}{n}\right]\binom{n(k+1)}{n} = \binom{n(k+1)}{n}\frac{nk!}{\prod_{i=1}^{k}(n!)}$$

$$= \frac{[n(k+1)]!}{n![n(k+1)-n]!}\frac{nk!}{\prod_{i=1}^{k}(n!)} = \frac{[n(k+1)]!}{n!(nk!)}\frac{nk!}{\prod_{i=1}^{k}(n!)}$$

$$= \frac{[n(k+1)]!}{n!\prod_{i=1}^{k}(n!)} = \frac{N!}{\prod_{i=1}^{k+1}(n!)}, \quad \text{where } N = [n(k+1)]$$

Thus, the theorem is also true for $p = k+1$ and hence true in general.

## 2.2 Mathematical Formulation of Algorithm for Unbalanced Cases: (where $n_i$'s are not all equal)

***Two Treatments***

As already seen in the balanced case, if we take $n_i$ = number **of variates** in the ith sample, the number of permutations for a two-sample case

$$= \sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i} \qquad \textbf{(2.1)}$$

Note that (2.1) assumes that $n = \min(n_1, n_2)$. **It is also true if we allow $n = n_1$ or $n_2$** since

$$\binom{a}{b} = 0 \text{ for } b > a.$$

***Three Treatments***

We observe that when we extend our **permutation to three treatments,** (2.1) becomes

**Theorem 2.2**

The number of partitionings of the elements of p samples **of unequal sizes** required for the distribution of the $W_N$ statistic is given by $\prod_{j=1}^{p}\binom{\sum_{r=1}^{j} n_r}{n_j}$

**Proof (By Induction)**

Again, we simply need to show that $\prod_{j=1}^{p}\binom{\sum_{r=1}^{j} n_r}{n_j} = \dfrac{N!}{\prod_{i=1}^{k}(n_i!)}$ = **number** of possible partitionings of N variates of p subsets, each of size $n_i$, which are equally **likely and** all the $n_i$'s are not equal.

Take p = 1: $\prod_{j=1}^{1}\binom{\sum_{r=1}^{j} n_r}{n_j} = \binom{n_1}{n_1} = \dfrac{n_1!}{n_1!} = \dfrac{N!}{\prod_{i=1}^{1}(n_i!)}$, where $N = n_1$ **for p = 1**

Take p = 2 $\prod_{j=1}^{2}\binom{\sum_{r=1}^{j} n_r}{n_j} = \binom{\sum_{r=1}^{1} n_r}{n_1}\binom{\sum_{r=1}^{2} n_r}{n_2} = \binom{n_1}{n_1}\binom{n_1+n_2}{n_2} = \binom{n_1+n_2}{n_2}$

$= \dfrac{(n_1+n_2)!}{n_1!n_2!} = \dfrac{N!}{\prod_{i=1}^{p}(n_i!)}$, where $N = n_1 + n_2$ **for p = 2**

Assume true for p = k

$$\prod_{j=1}^{k}\binom{\sum_{r=1}^{j} n_r}{n_j} = \frac{\sum_{r=1}^{k} n_r}{\prod_{i=1}^{k}(n_i!)}$$

Now, let us take p = k + 1

$$\prod_{j=1}^{k+1}\binom{\sum_{r=1}^{j} n_r}{n_j} = \binom{\sum_{r=1}^{k+1} n_r}{n_{k+1}}\left(\prod_{i=1}^{k}\left[\begin{matrix}\sum_{r=1}^{j} n_r \\ n_j\end{matrix}\right]\right)$$

$$= \binom{\sum_{r=1}^{k+1} n_r}{n_{k+1}} \frac{\left[\sum_{r=1}^{k} n_r\right]!}{\prod_{j=1}^{k}(n_j!)} = \frac{\left(\sum_{r=1}^{k+1} n_r\right)!}{\left[\sum_{r=1}^{k+1} n_r - n_{k+1}\right]! \, n_{k+1}!} \frac{\left[\sum_{r=1}^{k} n_r\right]!}{\prod_{j=1}^{k}(n_j!)}$$

$$= \frac{\left[\sum_{r=1}^{k} n_r\right]!}{\left[\prod_{r=1}^{k+1} n_r - n_{k+1}\right]!} \left\{ \frac{\left[\sum_{r=1}^{k+1} n_r\right]!}{n_{k+1}! \prod_{j=1}^{k}(n_j!)} \right\}$$

$$= \frac{\left[\sum_{r=1}^{k} n_r\right]!}{\left[\prod_{r=1}^{k} n_r\right]!} \left\{ \frac{\left[\sum_{r=1}^{k+1} n_r\right]!}{\prod_{j=1}^{k+1}(n_j!)} \right\}$$

$$= \frac{\left(\sum_{r=1}^{k+1} n_r\right)!}{\prod_{j=1}^{k+1}(n_j!)} = \frac{N!}{\prod_{j=1}^{k+1}(n_j!)}$$

This shows that the theorem is also true for $p = k + 1$ and hence we conclude that the theorem is true in general.

## 3. IMPLEMENTATION OF THE ALGORITHM

We now present an illustrative implementation of the algorithm to generate all the possible partitionings of the N variates into p subsets of size $n_i$, $i = 1(1)p$.

Let us examine a matrix of two variables, each with two variates, i.e. $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$. We expect to have $\frac{4!}{2!2!} = 6$ configurations

The partitionings are as follow:

| No | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Configuration | A C<br>B D | C A<br>B D | D C<br>B A | A B<br>C D | A C<br>D B | C A<br>D B |

Observe that configuration (1) is the original matrix, configurations (2) to (5) are configurations obtained by using the elements of the first column to interchange the elements of the second column, each in turn. Configuration (6) is obtained by interchanging the columns.

Let us examine a matrix of 2 variables, each with 3 variates, i.e. $\begin{pmatrix} A & D \\ B & E \\ C & F \end{pmatrix}$.

We expect to have $\frac{6!}{3!3!} = 20$ configurations

The configurations are as follow:

| No | Config. | No | Config. | No | Config. |
|---|---|---|---|---|---|
| | A D | | A C | | D A |
| 1 | B E | 8 | B E | 15 | B E |
| | C F | | D F | | F C |
| | D A | | A D | | E D |
| 2 | B E | 9 | B C | 16 | B A |
| | C F | | E F | | F C |
| | E D | | A D | | A B$^{(B,C)}$ |
| 3 | B A | 10 | B E | 17 | D C |
| | C F | | F C | | E F |
| | F D | | D A$^{(A,B)}$ | | A B |
| 4 | B E | 11 | E B | 18 | D E |
| | C A | | C F | | F C |
| | A B | | D A | | A D |
| 5 | D E | 12 | F E | 19 | E B |
| | C F | | C B | | F C |
| | A D | | E D | | D A |
| 6 | E B | 13 | F A | 20 | E B |
| | C F | | C B | | F C |
| | A D | | D A$^{(A,C)}$ | | |
| 7 | F E | 14 | B C | | |
| | C B | | E F | | |

The elements in the parenthesis were those used for interchange of elements of second column

Observe that configuration (1) is the original matrix, configurations (2) to (10) are obtained by using the elements of the first column to interchange the elements of the second column, one at a time. Configurations (11) to (19) are obtained by using 2 elements of the first column to interchange the elements of the second column. The elements used (A,B), (A,C) and (B,C) are indicated in parenthesis on top of configurations 11, 14, and 17. Lastly, configuration (20) is obtained by interchanging the columns.

## 4. TEST PROCEDURE

The multiple comparison test procedure for $P \geq 2$ samples (treatments) may now be formulated as stated in Ogbonmwan (1983) as follows:

1. Compute $W_N$: $W_N = n^{1/2} A_N^{-1}\left[\max h_N(X_i) - \min h_N(X_j)\right]$, i = 1(1)p

2. Compute the value of $W_N$ in (1), then, consider the probability $P(W_N)$ of the different values of $W_N$. Pick out the value $W_{N,P}(\alpha_o)$ of $W_N$ corresponding to the preassigned $\alpha_o$-level of significance under Ho : $F_1 = F_2 = F_3 = \ldots = F_p$, F(x) is assumed to be continuous (so that the probability of ties may be ignored).
$h_N(X_i)$, i = 1(1)p, are all rank order statistics, it follows that the permutation distribution of $W_N$ derived in the manner described above will surely give the exact null distribution of $W_N$ for large $n_i$'s.

3. To check whether the p treatments come from the same population, we regard the p treatments to be significantly different if

$$n^{1/2} A_N^{-1}\left[\max h_N(X_i) - \min h_N(X_j)\right] \geq W_{N,P}(\alpha_o),\ i = 1(1)p$$

4. If the null hypothesis of "no difference" between the P treatments is rejected, the following procedure is used to verify the treatment(s) that are significantly different from the others. Test using steps (1), (2) and (3) above, the sets of (p-1) treatments; sets of (p-2) treatments, ..., sets of 2 treatments, all taken from the p treatments.
If any set of $k \leq p$ treatments is found not to be significantly different, then any subset of the k treatments are also not significantly different.

## 5. DISTRIBUTION OF THE $W_N$ STATISTIC

By generating the partitionings using the algorithm described above, and computing the value of $W_N$ for each partitioning, Tables 1A and 1B give the distribution of the $W_N$ statistic with two and three treatments respectively.

**Table 1A:** **Exact Distribution of $W_N$ Statistic for p treatments, k variates (p = 2)**

| p = 2, k = 2 | | p = 2, k = 3 | | p = 2, k = 4 | | p = 2, k = 5 | | p = 2, k = 6 | |
|---|---|---|---|---|---|---|---|---|---|
| $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ |
| 0 | 0.333333 | 0.33806 | 0.3 | 0 | 0.114285 | 0.15569 | .15873 | 0 | .06277 |
| 1.26491 | 0.333333 | 1.01418 | 0.3 | 0.43643 | 0.2 | 0.46709 | .150793 | .23652 | .119047 |
| 2.52982 | 0.333333 | 1.6903 | 0.2 | 0.87287 | 0.2 | 0.77849 | .142857 | .47304 | .119047 |
| | | 2.36643 | 0.1 | 1.3093 | 0.142857 | 1.08989 | .126984 | .70957 | .110389 |
| | | 3.04255 | 0.1 | 1.74574 | 0.142857 | 1.40129 | .111111 | .94609 | .103896 |
| | | | | 2.18217 | 0.085714 | 1.71269 | .087301 | 1.18262 | .090909 |
| | | | | 2.61861 | 0.057142 | 2.02409 | .071428 | 1.41914 | .084415 |
| | | | | 3.05505 | 0.028571 | 2.33549 | .055555 | 1.65567 | .069264 |
| | | | | 3.49148 | 0.028571 | 2.64689 | .039682 | 1.89219 | .060606 |
| | | | | | | 2.95829 | .023809 | 2.12872 | .047619 |
| | | | | | | 3.26969 | .015873 | 2.36524 | .038961 |
| | | | | | | 3.58109 | .007936 | 2.60177 | .028138 |
| | | | | | | 3.89249 | .007936 | 2.83829 | .023809 |
| | | | | | | | | 3.07482 | .015151 |
| | | | | | | | | 3.31134 | .010822 |
| | | | | | | | | 3.54787 | .006493 |
| | | | | | | | | 3.78439 | .004329 |
| | | | | | | | | 4.02092 | .002164 |
| | | | | | | | | 4.25744 | .002164 |

**Table 1A (Contd.): Exact Distribution of $W_N$ Statistic for p treatments, k variates**

| p = 2, k = 7 | | p = 2, k = 8 | | p = 2, k = 9 | | p = 2, k = 10 | | p = 2, k = 10 (Contd.) | |
|---|---|---|---|---|---|---|---|---|---|
| $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ |
| .09376 | .098484 | 0 | .04087 | .06424 | .068572 | 0 | .029487 | 4.49694 | .000324 |
| .28128 | .096736 | .15339 | .080652 | .19274 | .06812 | .10968 | .058715 | 4.60663 | .000238 |
| .4688 | .094405 | .30678 | .080031 | .32124 | .067132 | .21936 | .058368 | 4.71631 | .000162 |
| .65633 | .090326 | .46017 | .077544 | .44974 | .065734 | .32904 | .057492 | 4.82599 | .000119 |
| .84385 | .085081 | .61357 | .075524 | .57824 | .063965 | .43872 | .056571 | 4.93567 | .000075 |
| 1.03137 | .079254 | .76696 | .071639 | .70674 | .061661 | .5484 | .055153 | 5.04535 | .000054 |
| 1.21889 | .072843 | .92035 | .068376 | .83524 | .059111 | .65808 | .053681 | 5.15503 | .000032 |
| 1.40642 | .065268 | 1.07375 | .063558 | .96373 | .056273 | .76777 | .051787 | 5.26472 | .000021 |
| 1.59394 | .058275 | 1.22714 | .059518 | 1.09223 | .053146 | .87745 | .049892 | 5.3744 | .00001 |
| 1.78146 | .050699 | 1.38053 | .054079 | 1.22073 | .049773 | .98713 | .047597 | 5.48408 | .00001 |
| 1.96899 | .043706 | 1.53393 | .049572 | 1.34923 | .0464 | 1.09681 | .045378 | | |
| 2.15651 | .036713 | 1.68732 | .044133 | 1.47773 | .04278 | 1.20649 | .042824 | | |
| 2.34403 | .030885 | 1.84071 | .039627 | 1.60623 | .039243 | 1.31618 | .040366 | | |
| 2.53155 | .024475 | 1.9941 | .034343 | 1.73472 | .035664 | 1.42586 | .037682 | | |
| 2.71908 | .019813 | 2.1475 | .030147 | 1.86322 | .032167 | 1.53554 | .035138 | | |
| 2.9066 | .015151 | 2.30089 | .025485 | 1.99172 | .028753 | 1.64522 | .03241 | | |
| 3.09412 | .011655 | 2.45428 | .021911 | 2.12022 | .025586 | 1.7549 | .029888 | | |
| 3.28165 | .008741 | 2.60768 | .018026 | 2.24872 | .022418 | 1.86458 | .027246 | | |
| 3.46917 | .00641 | 2.76107 | .015073 | 2.37722 | .01958 | 1.97427 | .024821 | | |
| 3.65669 | .004079 | 2.91446 | .011965 | 2.50572 | .016906 | 2.08395 | .022353 | | |
| 3.84421 | .002913 | 3.06786 | .00979 | 2.63421 | .014479 | 2.19363 | .020134 | | |
| 4.03174 | .001748 | 3.22125 | .007459 | 2.76271 | .012217 | 2.30331 | .017883 | | |
| 4.21926 | .001165 | 3.37464 | .005905 | 2.89121 | .010324 | 2.41299 | .015912 | | |
| 4.40678 | .000582 | 3.52803 | .004351 | 3.01971 | .008515 | 2.52267 | .013953 | | |

| p = 2, k = 7 | | p = 2, k = 8 | | p = 2, k = 9 | | p = 2, k = 10 | | p = 2, k = 10 (Contd.) | |
|---|---|---|---|---|---|---|---|---|---|
| $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ |
| 4.59431 | .000582 | 3.68143 | .003418 | 3.14821 | .007034 | 2.63236 | .012243 | | |
| | | 3.83482 | .002331 | 3.27671 | .005676 | 2.74204 | .010586 | | |
| | | 3.98821 | .001709 | 3.40521 | .004566 | 2.85172 | .009168 | | |
| | | 4.14161 | .001087 | 3.5337 | .003578 | 2.9614 | .007794 | | |
| | | 4.295 | .000777 | 3.6622 | .002838 | 3.07108 | .006657 | | |
| | | 4.44839 | .000466 | 3.7907 | .002139 | 3.18076 | .005574 | | |
| | | 4.60179 | .00031 | 3.9192 | .001645 | 3.29045 | .004687 | | |
| | | 4.75518 | .000155 | 4.0477 | .001234 | 3.40013 | .003853 | | |
| | | 4.90857 | .000155 | 4.1762 | .000904 | 3.50981 | .003193 | | |
| | | | | 4.3047 | .000617 | 3.61949 | .002565 | | |
| | | | | 4.43319 | .000452 | 3.72917 | .002089 | | |
| | | | | 4.56169 | .000287 | 3.83885 | .001645 | | |
| | | | | 4.69019 | .000205 | 3.94854 | .001309 | | |
| | | | | 4.81869 | .000123 | 4.05822 | .001006 | | |
| | | | | 4.94719 | .000082 | 4.1679 | .00079 | | |
| | | | | 5.07569 | .000041 | 4.27758 | .000584 | | |
| | | | | 5.20418 | .000041 | 4.38726 | .000454 | | |

**Table 1B: Exact Distribution of $W_N$ Statistic for p treatments, k variates (p = 3)**

| p = 3, k = 2 | | p = 3, k = 3 | | p = 3, k = 4 | | p = 3, k = 5 | | p = 3, k = 5 (Contd.) | |
|---|---|---|---|---|---|---|---|---|---|
| $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ |
| 0 | 066666 | 0 | .007142 | 0 | .005541 | 0 | .002418 | 3.51933 | 008063 |
| .82807 | .133333 | .44721 | .064285 | .28968 | .026839 | .20701 | .014992 | 3.62284 | .006913 |
| 1.24211 | 133333 | .67082 | .05 | .43452 | .026666 | .31052 | .014572 | 3.72635 | .005605 |
| 1.65615 | 133333 | .89442 | .05 | .57936 | .027878 | .41403 | .014287 | 3.82986 | .004471 |
| 2.07019 | .133333 | 1.11803 | .107142 | .7242 | .049177 | .51754 | .028368 | 3.93337 | .003718 |
| 2.48423 | 2 | 1.34164 | .092857 | .86904 | .048831 | .62105 | .027337 | 4.03688 | .002886 |
| 2.89827 | 133333 | 1.56524 | .085714 | 1.01388 | .045367 | .72456 | .026988 | 4.14039 | .002101 |
| 3.31231 | 066666 | 1.78885 | .103571 | 1.15873 | .063549 | .82807 | .038826 | 4.2439 | .001538 |
| | | 2.01246 | .078571 | 1.30357 | .057489 | .93158 | .037375 | 4.34741 | .001062 |
| | | 2.23606 | .064285 | 1.44841 | .056969 | 1.03509 | .03571 | 4.45092 | .000713 |
| | | 2.45967 | .078571 | 1.59325 | .064415 | 1.1386 | .045462 | 4.55443 | .000444 |
| | | 2.68328 | .057142 | 1.73809 | .06251 | 1.24211 | .042727 | 4.65794 | .000285 |
| | | 2.90688 | .05 | 1.88293 | .055064 | 1.34562 | .040959 | 4.76145 | .000158 |
| | | 3.13049 | .046428 | 2.02777 | .060432 | 1.44913 | .047175 | 4.86496 | .000079 |
| | | 3.3541 | .035714 | 2.17262 | .050562 | 1.55264 | .044114 | 4.96847 | .000039 |
| | | 3.5777 | .017857 | 2.31746 | .047272 | 1.65615 | .041038 | 5.07198 | .000015 |
| | | 3.80131 | .007142 | 2.4623 | .043982 | 1.75966 | .04543 | 5.17549 | .000007 |
| | | 4.02492 | .003571 | 2.60714 | .038614 | 1.86317 | .041506 | | |
| | | | | 2.75198 | .031861 | 1.96668 | .038786 | | |
| | | | | 2.89682 | .030649 | 2.07019 | .040031 | | |
| | | | | 3.04166 | .023549 | 2.1737 | .036693 | | |
| | | | | 3.1865 | .00277 | 2.27721 | .032935 | | |
| | | | | 3.18651 | .018008 | 2.38072 | .025379 | | |
| | | | | 3.33135 | .017316 | 2.48423 | .033411 | | |
| | | | | 3.47619 | .014025 | 2.58774 | .029573 | | |
| | | | | 3.62103 | .011082 | 2.69125 | .026671 | | |
| | | | | 3.76587 | .008138 | 2.79476 | .02239 | | |
| | | | | 3.91071 | .004848 | 2.89827 | .019274 | | |
| | | | | 4.05555 | .003463 | 3.00178 | .017839 | | |
| | | | | 4.20039 | .001731 | 3.10529 | .015238 | | |
| | | | | 4.34524 | .000865 | 3.2088 | .013018 | | |
| | | | | 4.49008 | .000346 | 3.31231 | .011591 | | |
| | | | | 4.63492 | .000173 | 3.41582 | .009768 | | |

**Table 1B (Cont): Exact Distribution of $W_N$ Statistic for p treatments, k variates (p=3)**

| p = 3, k = 6 | | p = 3, k = 6 (Contd.) | |
|---|---|---|---|
| $W_N$ | $P(W_N)$ | $W_N$ | $P(W_N)$ |
| 0 | .001517 | 3.06886 | .012686 |
| .07868 | .000024 | 3.14755 | .011367 |
| .15737 | .008889 | 3.22624 | .010336 |
| .23606 | .008828 | 3.30493 | .009101 |
| .31475 | .008928 | 3.38362 | .007931 |
| .39344 | .017304 | 3.46231 | .007135 |
| .47213 | .017184 | 3.541 | .006104 |
| .55082 | .016924 | 3.61969 | .005307 |
| .62951 | .024782 | 3.69838 | .004599 |
| .7082 | .024172 | 3.77706 | .003915 |
| .78688 | .023976 | 3.85575 | .003285 |
| .86557 | .030569 | 3.93444 | .002814 |
| .94426 | .02988 | 4.01313 | .002319 |
| 1.02295 | .029051 | 4.09182 | .00193 |
| 1.10164 | .034823 | 4.17051 | .001575 |
| 1.18033 | .033433 | 4.2492 | .001284 |
| 1.25902 | .032677 | 4.32789 | .001029 |
| 1.33771 | .03677 | 4.40658 | .000814 |
| 1.4164 | .03545 | 4.48526 | .000076 |
| 1.49508 | .017135 | 4.48527 | .000545 |
| 1.49509 | .01677 | 4.56395 | .000478 |
| 1.57377 | .037112 | 4.64264 | .000354 |
| 1.65246 | .035037 | 4.72133 | .000262 |
| 1.73115 | .033641 | 4.80002 | .000186 |
| 1.80984 | .035279 | 4.87871 | .000131 |
| 1.88853 | .033405 | 4.9574 | .000088 |
| 1.96722 | .03137 | 5.03609 | .000059 |
| 2.04591 | .03239 | 5.11478 | .000037 |
| 2.1246 | .03002 | 5.19347 | .000024 |
| 2.20329 | .02825 | 5.27215 | .000013 |
| 2.28197 | .028125 | 5.35084 | .000007 |
| 2.36066 | .026078 | 5.42953 | .000004 |
| 2.43935 | .023984 | 5.50822 | .000002 |
| 2.51804 | .023629 | 5.58691 | 0 |
| 2.59673 | .021423 | 5.6656 | 0 |
| 2.67542 | .019703 | 5.74429 | 0 |
| 2.75411 | .018769 | | |
| 2.8328 | .016996 | | |
| 2.91149 | .01527 | | |
| 2.99017 | .006373 | | |
| 2.99018 | .007998 | | |

## 6. CONCLUSION

In Ogbonmwan (1983), the $W_N$ statistic was introduced, the asymptotic distribution was proved but there was no theoretical proof for the exact distribution. The exact distribution was however considered via a permutation approach which was not implemented empirically. In this work, the exact probability distribution table has been provided for small sample sizes. The mathematical formulation of the algorithm for generating the configurations are fully discussed and explained in Section 2. Theorems concerning the number of partitionings of the elements of p samples for equal and unequal sizes required for the distribution of the $W_N$ statistic were stated and proved. Furthermore, the algorithms were implemented giving rise to the statistical table for the exact distribution of $W_N$ statistic for p treatments and k variates: $p = 2, 3; k = 2, 3, 4, \ldots, 10$. These have never been given elsewhere.

The results of this work conclude the introduction of the $W_N$ as a test statistic for handling p-sample problems

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