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CONVERGENT EMBEDDED EXPLICIT RUNGE –KUTTA (CEERK)METHOD OF ORDER 4

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ABSTRACT:

Some Mathematical Modeling in the Physical sciences have often led to a system of ordinary differential equation of the first order that is dependent on the time variable t.

Authors such as [1], [2], [4], [5] and [3] have contributed greatly in the numerical solution of such a differential equation. In this direction, we shall propose a Convergent Embedded Explicit Runge-kutta (CEERK) Method that can cope with the differential equation. Numerical solution of the differential equation obtained by the proposed CEERK Method compares favourably with the Classical Explicit Runge-kutta Method of order 4.

Keywords: Differential Equation, Runge-kutta, Embedded,Time, Stepsize.

1.0: INTRODUCTION

Consider the time dependent first order ordinary differential equation:

$$\frac{dy(t)}{dt} = f(t, y(t)) \tag{1.1}$$

where $\frac{dy(t)}{dt}$ is the rate of change of quantity y(t) with respect to time, t.

At the initial time, t_o we shall assume that

$$y(t_o) = y_o \tag{1.2}$$

Equations (1.1) and (1.2) combined together is called an initial value problem. Now, we shall seek the numerical (approximate) solution of such initial value problem.

According to Runge (1895) and Kutta (1901), an s Stage Explicit Runge-kutta method is defined by

$$y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i k_i \qquad , n = o,1,2,.... \tag{1.3}$$

$$k_i = f\left(t_n + c_i h_n, y_n + h_n \sum_{j=1}^{i-1} a_{ij} k_j\right) ... \quad (1.4)$$

with the constraint

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \qquad i = 1,2,...s \quad (1.5)$$

and the step size

$$h_n = t_{n+1} - t_n, \qquad n = o,1,2,... \quad (1.6)$$

where the k_i's, c_i's and b_i's are the slopes, nodes and the weights of the method respectively.

For each integer n, y_n is interpreted as an approximation to the value $y(t_n)$, of the theoretical solution of the initial value problem. Verner [4] showed that, for appropriate choice of parameters a_{ij} and nodes c_i in (1.4), two approximations are obtained. The two approximations are

$$y_{p-1,n+1} = y_n + h_n \sum_{i=1}^{s} b_{p-1,i} k_i \quad (1.7)$$

and

$$y_{p,n+1} = y_n + h_n \sum_{i=1}^{s} b_{p,i} k_i$$

with the error estimate obtained from the difference between $y_{p-1,n+1}$ and $y_{p,n+1}$. This error estimate [4] helps us to decide if y_{n-1} is a sufficiently accurate approximation to $y(t_n)$ and to select a trial stepsize for the subsequent approximation.

Lambert [5] and Fatunla [6] showed that a lower order explicit Runge-kutta method can be derived by a Taylor series expansion. Butcher [2], Hairer et al [7] suggested a better way of deriving order equations for explicit Runge kutta methods of higher order by the concept of labelled (rooted) trees. Verner [4] showed that the order equations can be partitioned into special classes of problems: Quadrature problem and the system of linear constant coefficient differential equations.

Motivated by the improved efficiency that may result from strategies for changing order, this paper develops a convergent embedded Explicit Runge-kutta method a Runge-Kutta method of lower order is embedded in one of the higher order if the two methods use the same functions evaluations.

2.0: DERIVATION OF CEERK METHOD

Definition 2.1: For an integer, s and index, I a set of embedded Explicit Runge-kutta methods is a set of s functions evaluation given by (1.4) together with linearly independent approximations.

$$y_{p-1,n+1} = y_n + h_n \sum_{i=1}^{s} b_{p-1,i} k_i \qquad , p \in I, \ldots \tag{2.1.1}$$

satisfying

$$y_{p,n+1} = y(t_n + h_n t_n, y_n) + 0\left(h_n^{p+1}\right) \tag{2.1.2}$$

Theorem 2.2:

For each (s, p) Explicit Runge-kutta method, there exists a set of embedded s Explicit Runge-kutta methods (s, I) with I containing s positive integers in [1,p].

Proof: see [1].

Butcher [2] represented (s, p) Explicit Runge-kutta methods by the tableau

$$\begin{array}{c|ccccc}
c_1 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
c_4 & a_{41} & a_{42} & a_{43} & \cdots & \\
\vdots & \vdots & \vdots & \vdots & & \\
c_s & a_{s1} & a_{s2} & a_{s3} & \cdots & a_{s;s-1} \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_s
\end{array} \tag{2.3}$$

The parameters of (2.3) satisfy the order equations of all orders (see [1], [2] and [3]). The CEERK method of order 4 shall be derived.

Now, for $p \in \{1, 2, 3, 4\}$ we consider the following order equations

$$\sum_{i=1}^{4} b_i c_i^k - \frac{1}{k+1} = 0, k = 0,1,2,3 \quad (2.4)$$

$$\sum_{j=1}^{3}\sum_{i=1}^{4} b_i a_{ij} c_j^k - \frac{1}{(k+1)(k+2)} = 0, \qquad k = 0,1,2 \quad (2.5)$$

$$\sum_{l=1}^{2}\sum_{j=1}^{3}\sum_{i=1}^{4} b_i a_{ij} a_{jl} c_i^k - \frac{1}{(k+1)(k+2)(k+3)}, = 0, \quad k = 0,1 \quad (2.6)$$

$$\sum_{j=1}^{3}\sum_{i=1}^{4} b_i c_i^{q-1} a_{ij} c_j^{r-1} - \frac{1}{r(q+r)} = 0, \ q + r \le 4 \quad (2.7)$$

$$\sum_{K=1}^{2}\sum_{j=1}^{3}\sum_{i=1}^{4} b_i c_i^{q-1} a_{ij} c_j^{r-1} a_{jk} c_h^{s-1} - \frac{1}{(q+r+s)(r+s)s} = 0, \quad q + r + s \le 4 \quad (2.8)$$

To achieve our target, we shall adopt the following simplifying assumptions:

$$\sum_{j=1}^{i-1} a_{ij} c_j^k = \frac{c_i^{k+1}}{k+1}, i = 1(1)4, \ k \le 4 \quad (2.9)$$

and

$$\sum_{i=1}^{4} b c_i^k a_{ij} = \frac{b_j\left(1 - c^{k+1}\right)}{k+1}, \quad j = 1(1)4 \qquad k \le 4 \quad (2.10)$$

where k is the degree of precision.

We noticed that each equation is linear in the vector of weights $\{b_i\}$. Hence, the CEERK method of order 4 is obtained if b_i is the solution of the linear system of equations:

$$M_4 \, b = T_4 \quad (2.11)$$

where M_4 is a 4th order matrix determined by the parameter $\{c_i\}$, and T_4 is the vector arising from the coefficients of elementary differentials in the Taylor series expansion of $y(t_n + h_n, t_n, y_n)$.

If we impose a further constraint [8] on the nodes that $c_4 = 1$, we find that the parameters satisfy one set of simplifying conditions.

$$\sum_{j=1}^{i-1} a_{ij} c_j^k = \frac{c_i^{k+1}}{k+1}, i = 1(1)4, \quad k = 0,1 \quad (2.12)$$

$$\sum_{i=j+1}^{4} b_i a_{ij} = b_j(1-c_j),\ j = 1(1)4 \tag{2.13}$$

Hence, for $c_4 = 1$, then $b_4 \neq o$. Furthermore, imposing the following constraints on the nodes

$$c_2 + c_3 = c_4 \tag{2.14}$$

With $c_2 \neq c_3$, $c_2 \neq 1$ or $c_3 \neq 1$, $c_2 \neq o$ or $c_3 \neq o$, $c_2 \neq 2/3$, we obtain the value of the nodes:

$$\{c_1 = o, c_2 = 1/4, c_3 = 3/4, c_4 = 1\}$$

which are distinct. It follows that

$$\det[M_4] = 1/512 \tag{2.15}$$

Since the matrix M_4 is nonsingular then the solution, $\{b_1 = 7/18, b_2 = 1/9, b_3 = 4/9, b_4 = 1/18$ of (2.11) is a column vector $b = \{b_i / i = 1(1)4\}$ obtained from

$$b = 152.T_4.\mathrm{Adj}(M_4) \tag{2.16}$$

Consequently, other parameters $\{a_{ij}\}$ are obtained:

$$a_{21} = \frac{1}{4},\quad a_{31} = -3/4,\ a_{32} = 3/2,\quad a_{41} = 5, a_{42} = -6,\ \text{ and } a_{43} = 2$$

According to [1], there exists an Explicit Runge-kutta method of order 3 embedded in the method of order 4 with the same function evaluations.
Hence, we have the Convergent Embedded Explicit Runge-kutta (CEERK) method of order 4 described by the tableau below:

0				
$\frac{1}{4}$	$\frac{1}{4}$			
$\frac{3}{4}$	$-\frac{3}{4}$	$\frac{3}{2}$		
1	5	-6	2	
b	$\frac{7}{18}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{1}{18}$
$\hat{b}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	

3.0: ERROR ESTIMATE

The error estimate is obtained by the method of embedding (see [1]). Hence, the error estimate for the CEERK method is defined as:

$$\left|y_{n+1,4} - y_{n+1,3}\right| = h_n\left|\sum_{i=1}^{4} b_{i,4}k_i - \sum_{i=1}^{3} b_{i,3}k_i\right| = h_n\left|\sum_{i=1}^{4} b_i k_i - \sum_{i=1}^{3} \hat{b}_i k_i\right|$$

where

$$k_i = f\left(t_n + c_i h_n, y_n + h_n \sum_{j=1}^{i-1} a_{ij} k_j\right), \qquad i = 1(1)4 \qquad (3.1)$$

4.0: NUMERICAL EXPERIMENT:

Consider the problem [6]:

$$\frac{dy(t)}{dt} = -10(y(t)-1)^2, y(o) = 2 \text{ with } t\,\varepsilon[o,1] \text{ and } h_n = h = 0.01$$

Exact Solution: $y(t) = \dfrac{2+10t}{1+10t}$

Numerical Result:
For 10 iterations performed, the results are shown in the table below:

Table 1

Number of Iterations	CEERK Method of order 4	Classical Runge-kutta Method of order 4	Exact Solution y(t)	Error Estimate By Method of Embedding
1	1.8813341	1.8790079	1.9090909	6.4957×10^{-3}
2	1.7935841	1.8079855	1.8333333	16.2539×10^{-3}
3	1.7344121	1.7475821	1.7692307	7.5100×10^{-4}
4	1.6833933	1.6955818	1.7142857	3.3580×10^{-4}
5	1.6391507	1.6503451	1.6666667	4.8770×10^{-4}
6	1.6003218	1.6106330	1.6250000	4.0140×10^{-4}
7	1.5659663	1.5754917	1.5882352	3.3344×10^{-4}
8	1.5353504	1.5441750	1.5555555	2.8150×10^{-4}
9	1.5078929	1.5160907	1.5263157	2.3930×10^{-4}
10	1.4831273	1.4907629	1.500000	2.0500×10^{-4}

5.0: CONCLUSION

The convergent Embedded Explicit Runge-kutta (CEERK) Method compares favourably with the existing Classical Runge-kutta method of order 4.
(See table 1).

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