J. Nig. Ass. Math. Phys.
Vol. 5 2001

AN ITERATIVE METHOD FOR SIMULTANEOUS INCLUSION OF POLYNOMIAL ZEROS.

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ABSTRACT

Let $P_n(z)\,; n > 1$ be a polynomial of degree n. Methods that not only enclose , but also isolate the n zeros in a suitable interval have been proposed. See for examples, Petkovic [7] , Gargantini [1,2,3], Gargantini and Henrici [4], Petkovic and Carstensen [8] , etc. These methods have the disadvantages of being complicated , but they possess as one of their features the automatic determination of bounds for all zeros. In this paper, we present an interval method and its analysis of the order of convergence. However, the analysis is presented in circular interval arithmetic introduced by Gargantini and Henrici [4, 5] . We illustrate convergence of the method by numerical examples. Extension of the approach to rectangular arithmetic is analogous.

1. Introduction

Define the complex circular interval

$$Z = \{z : |\, z - a \,| \le r;\, a \in C, r \in R, r => 0\} = \{z; r\} \tag{1.1}$$

which is precisely a disk with radius r and centre at a. Denote this as

$$a = mid(Z)\,;\, r = rad(Z) \tag{1.2}$$

The arithmetic operations $\{+,-,\times,/\}$ on the complex circular interval in (1.1) are found in Petkovic [7] and Gargantini [1]. To fix ideas we adopt the following notations. Let

$$\begin{gathered} z_j = mid(Z_j)\,;\, r_j = rad(Z_j)\,;\, r = \max_j \{r_j\};\, i, j = 1(1)n \\ e_j = z_j - \lambda_j\,;\, e = \max_j \{e_j\};\, \rho = \min_{\substack{i,j \\ i \ne j}} \{|z_i - z_j| - r_j\} \end{gathered} \tag{1.3}$$

where $\{\lambda_j\}_{j=1}^{n}$ are the simple zeros of the polynomial $P_n(z) = \sum_{j\;0}^{n} a_j z^j\,;\, a_n \ne 0, a_j \in \mathbb{C}$, which zeros we seek to enclose in circular disks

$Z_j = \{z_j ; r_j\}$ of insignificant radius r_j. From the third order simultaneous point iteration method

$$Z_j^{(s+1)} = z_j^{(s)} - F(z_j^{(s)})\left(1 - \sum_{i=1;j\neq i}^{n}\left(\frac{F(z_i^{(s)})}{z_j^{(s)} - z_i^{(s)}}\right)\right) ; \ s = 0,1,2,...;$$

$$F(z_j^{(s)}) = \frac{P_n(z_i^{(s)})}{\prod_{k=1;k\neq i}^{n} (z_i^{(s)} - z_k^{(s)})}$$

of Milovanovic [9] an inclusion method is derivable by the replacement of the point $\sum_{i=1;j\neq i}^{n}\left(\frac{1}{z_j^{(s)} - z_i^{(s)}}\right)$ with $\sum_{i=1;j\neq i}^{n}\left(\frac{1}{z_j^{(s)} - Z_i^{(s)}}\right)$; $s = 0,1,2,...$ to obtain the interval method

$$Z_j^{(s+1)} = z_j^{(s)} - F(z_j^{(s)})\left(1 - \sum_{i=1;j\neq i}^{n}\left(\frac{F(z_i^{(s)})}{z_j^{(s)} - Z_i^{(s)}}\right)\right) ; \ s = 0,1,2,... \qquad (1.4)$$

The replacement is by no means trivial and its implementation is guided by the rules of interval arithmetic , see Petkovic[7]. A variant is

$$Z_i^{(s+1)} = z_j^{(s)} - F(z_j^{(s)})\left(1 - \left\{\sum_{i=1;j\neq i}^{n}\left(\frac{F(z_i^{(s)})}{z_j^{(s)} - z_i^{(s)}}\right); \frac{r^{(s)} \sum_{i=1;j\neq i}^{n} |F(z_i^{(s)})|}{(\rho^{(s)})^2}\right\}\right) ; \ s = 0,1,2,... \qquad (1.5)$$

derived by using the fact that

$$\sum_{i=1;j\neq i}^{n}\left(\frac{1}{z_j^{(s)} - Z_i^{(s)}}\right) \subset \left\{\sum_{i=1;j\neq i}^{n}\left(\frac{1}{z_j^{(s)} - z_i^{(s)}}\right); \frac{(n-1)r}{\rho^2}\right\} \qquad (1.5a)$$

where s is an iteration index. From the inclusion relation

$$\sum_{i=1;j\neq i}^{n}\left(\frac{F(z_i^{(s)})}{z_j^{(s)} - Z_i^{(s)}}\right) \subset \left\{\sum_{i=1;j\neq i}^{n}\left(\frac{F(z_i^{(s)})}{z_j^{(s)} - z_i^{(s)}}\right); \frac{r^{(s)} \sum_{i=1;j\neq i}^{n} |F(z_i^{(s)})|}{(\rho^{(s)})^2}\right\} \qquad (1.5b)$$

$$z_i - F(z_i)\left(1 - \sum_{j=1;j\neq i}^{n}\left(\frac{F(z_j)}{z_i - Z_j}\right)\right)$$
$$= z_i - F(z_i)\left(1 - \left\{\sum_{j=1;j\neq i}^{n}\frac{(\overline{z_i} - \overline{z_j})F(z_j)}{|z_i - z_j|^2 - r_j^2};\ \sum_{j=1;j\neq i}^{n}\frac{r_j\,|F(z_j)|}{|z_i - z_j|^2 - r_j^2}\right\}\right) \tag{2.5}$$

by adopting the disk inversion arithmetic, see a section ahead. In this regard

$$Rad\left(z_i - F(z_i)\left(1 - \sum_{j=1;j\neq i}^{n}\left(\frac{F(z_j)}{z_i - Z_j}\right)\right)\right) = |F(z_i)| \sum_{j=1;j\neq i}^{n}\frac{r_j\,|F(z_j)|}{|z_i - z_j| - r_j^2}$$
$$= |F(z_i)| \sum_{j=1;j\neq i}^{n}\frac{r_j\,|F(z_j)|}{(\rho + 2r_j)\rho} \le \frac{|e_j|^2\left(\left(1 + \frac{r}{\rho}\right)^{n-1}\right)^2 r(n-1)}{\rho^2} \tag{2.6}$$

since $|e_j| = |z_j - \lambda_j| < r_j < r$ and with the requirement that $\rho > 3(n-1)r$ means then that

$$\left(\left(1 + \frac{r}{\rho}\right)^{n-1}\right)^2 \le \left(\left(1 + \frac{r}{3(n-1)}\right)^{n-1}\right)^2 \le e^{\frac{2}{3}} \le 2 \tag{2.7}$$

for an indefinite degree n of $P_n(z)$. Therefore

$$Rad\left(z_i - F(z_i)\left(1 - \sum_{j=1;j\neq i}^{n}\left(\frac{F(z_j)}{z_i - Z_j}\right)\right)\right) \le \frac{r^3(n-1)e^{\frac{2}{3}}}{(\rho)^2};\ e^{\frac{2}{3}} = 1.947734... < 2 \tag{2.8}$$

and the required bound follows. The above theorem still holds good for $\rho > 4(n-1)r$. Furthermore, if it is desirous to adopt the alternative definition $\rho_* = \min_{1\le i,j\le n} |z_i - z_j|$ in place of ρ , the above theorem still holds true. We propose now to prove the order of convergence theorem for (1.4).

Theorem 2.2

Let $Z_j^{(0)} = \left\{z_j^{(0)}; r_j^{(0)}\right\}$ be initial non-overlapping disks isolating the simple zeros

$\lambda_j \ ; j = 1(1)n$ respectively. If

$$\rho^{(0)} > 3(n-1)r^{(0)} \ ; n \geq 2 \tag{2.9}$$

then the sequence $\left\{Z_j^{(s)}\right\}_{j=1,s=0}^{n,\infty}$ of disks generated by the method (1.4) is such that

$$\begin{aligned} &\text{(a)} \ z_j^{(s)} \in Z_j^{(s)} \ ; s = 0,1,2,\dots ; j = 1(1)n \\ &\text{(b)} \ r^{(s+1)} \leq \frac{2(r^{(s)})^3(n-1)}{(\rho^{(0)} - \frac{7}{3}r^{(0)})^2} \end{aligned} \tag{2.9'}$$

so that the radii $\left\{r^{(s)}\right\}_{s=0}^{\infty} \to 0$ as s $\to \infty$ in the asymptotic sense expressed above. This means that the interval method (1.4) converges with order p=3.

Proof

Assume that $\lambda_j \in Z_j^{(s)} \ ; s = 0,1,2,\dots ; j = 1(1)n$ for some iteration index s. By inclusion monotonicity

$$\lambda_j \approx z_j^{(s)} - F(z_j^{(s)})\left(1 - \sum_{i=1,j\neq i}^{n}\left(\frac{F(z_i^{(s)})}{z_j^{(s)} - \lambda_i}\right)\right) \in z_j^{(s)} - F(z_j^{(s)})\left(1 - \sum_{i=1,j\neq i}^{n}\left(\frac{F(z_i^{(s)})}{z_j^{(s)} - Z_i^{(s)}}\right)\right) = Z_j^{(s+1)}; \quad s = 0,1,2,\dots \tag{2.10}$$

Because $\lambda_j \in Z_j^{(0)}$; $j = 1(1)n$ then applying mathematical induction on s in the above, the first assertion follows. For the second case

$$r_j^{(s+1)} = \mathrm{Rad}(Z_j^{(s+1)}) = \mathrm{Rad}\left(z_i^{(s)} - F(z_i^{(s)})\left(1 - \sum_{i=1,j\neq i}^{n}\left(\frac{F(z_i^{(s)})}{z_i^{(s)} - Z_j^{(s)}}\right)\right)\right) \leq \frac{2(n-1)(r^{(s)})^3}{(\rho^{(s)})^2} \tag{2.11}$$

using theorem (2.1), where of course

$$\rho^{(s)} = \min_{\substack{1 \leq j,i \leq n \\ j \neq i}} \left\{ |z_j^{(s)} - z_i^{(s)}| - r_j^{(s)} \right\}, r^{(s)} = \max_j \left\{ r_j^{(s)} \right\}, j = 1(1)n \tag{2.12}$$

For s = 0

$$r_j^{(1)} \leq \frac{2(n-1)(r^{(0)})^3}{(\rho^{(0)})^2}\,; j = 1(1)n \tag{2.13}$$

Hence

$$r^{(1)} \leq \frac{2(n-1)(r^{(0)})^3}{(\rho^{(0)})^2} < \frac{2\,r^{(0)}}{9(n-1)} \tag{2.14}$$

By geometric construction ,Gargantini [1], the disks $Z_1^{(1)}, Z_2^{(1)}, \ldots, Z_{n-1}^{(1)}, Z_n^{(1)}$ are disjoint if

$$\rho^{(0)} > | z_j^{(1)} - z_j^{(0)} | + 3\,r^{(1)} \tag{2.15}$$

In a worst case scenario $|z_j^{(1)} - z_j^{(0)}| \leq r^{(0)} + r^{(1)}$ thus

$$\rho^{(0)} > 4\,r^{(1)} + r^{(0)} > | z_j^{(1)} - z_j^{(0)} | + 3\,r^{(1)} \tag{2.16}$$

From the inequality

$$\rho^{(s+1)} \geq \rho^{(s)} - r^{(s)} - 3\,r^{(s+1)}\,; s = 0,1,2,\ldots \tag{2.17}$$

of Gargantini (1978), a particular instance is

$$\rho^{(1)} \geq \rho^{(0)} - r^{(0)} - 3\,r^{(1)} > 3(n-1)\,r^{(0)} - r^{(0)} - 3\,r^{(1)} \tag{2.18}$$

Thus from (2.14)

$$\rho^{(1)} \geq \left(3(n-1) - 1 - \frac{6}{9(n-1)} \right) r^{(0)} \geq 3(n-1) r^{((1} \left(3(n-1) - 1 - \frac{6}{9(n-1)} \right) \frac{3}{2} \geq 3(n-1) r^{(1)} \tag{2.19}$$

It is thus shown in general that by an inductive process

$$\rho^{(s)} > 3(n-1)\,r^{(s)}\,; s = 0,1,2,\ldots \tag{2.20}$$

Starting from the recursive inequality (2.17), $\rho^{(s)} \geq \rho^{(s-1)} - r^{(s-1)} - 3r^{(s)}; s=0,1,2,.$and using (2.14)

$$r^{(s)} < \frac{2}{9(n-1)} r^{(s-1)} < \frac{2}{9} r^{(s-1)} < \frac{1}{4} r^{(s-1)}; s = 0,1,2,... \qquad (2.21)$$

Then

$$\begin{aligned}
\rho^{(s)} &\geq \rho^{(s-1)} - r^{(s-1)} - 3\,r^{(s)} > \rho^{(s-1)} - r^{(s-1)}\left(1+\frac{3}{4}\right) \\
&\geq \rho^{(s-2)} - r^{(s-2)} - 3\,r^{(s-1)} - r^{(s-1)}\left(1+\frac{3}{4}\right) > \rho^{(s-2)} - r^{(s-2)}\left(1+\frac{3}{4}+\frac{1}{4}\left(1+\frac{3}{4}\right)\right) \\
&> \rho^{(s-3)} - r^{(s-3)} - 3\,r^{(s-2)} - r^{(s-2)}\left(1+\frac{4}{4}+\frac{4}{4^2}-\frac{1}{4^2}\right) \\
&> \rho^{(s-3)} - r^{(s-3)}\left(1+4\left(\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}\right)-\frac{1}{4^3}\right) \qquad (2.22)
\end{aligned}$$

$$> \rho^{(0)} - r^{(0)}\left(1+4\left(\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+...+\frac{1}{4^m}\right)-\frac{1}{4^m}\right) > \rho^{(0)} - \frac{7}{3} r^{(0)}$$

This in place in (2.11), the convergence order of the iterative method (1.4) is p=3. Theorem (2.2) is therefore established.

3. A COMPARISON OF THE METHOD (1.4) WITH THE PARALLEL NEWTON'S METHOD AND PARALLEL LAGUERRE'S METHOD.

The method (1.4) is also a parallel method in the sense that the computation at every iteration to realize a new disk isolating a zero can be carried out simultaneously and independently of all other disks. The comparisons shall be in the sense of Gargantini [3]. To compare the three procedures, parallel Method (1.4), parallel Newton's and parallel Laguerre's method we assume that:

(1) the initial disk $Z_j^{(0)}; j = 1(1)n$ represented as $\Gamma_j^{(0)}$ in Gargantini [3] is the same for the methods.

(2) we are considering the iteration on a disk at first iteration step

From these, a conclusive statement on the complexities of the algorithms after an M iterations can easily be inferred. The relevant details of the comparison of the parallel Newton's and parallel Laguerre's method are in Gargantini [3]. for the purpose of including (1.4) in the comparison we take the case

$$\rho^{(0)} > 6(n-1)\,r^{(0)}; n \geq 2 \qquad (3.1)$$

for which the sequence $\left\{r^{(s)}\right\}_{s=0}^{\infty}$ converges to zero in the three methods in the long run on the iteration index s. In what follows we establish a relation on the dependence of the number of iterations required to reach a given tolerance η for two cases of different initial separation for the starting disks $Z_j^{(0)}\ ;\ j = 1(1)n$.

Theorem 3.1
Let

(a) $r^{(0)} \le 1 \le \dfrac{\rho^{(0)}}{6(n-1)}\ ; n \ge 2$

(b) $1 < r^{(0)} < \dfrac{\rho^{(0)}}{6(n-1)}\ ; n \ge 2$

Then in each case of (a) or (b), the tolerance η is attained in $M_{(1.4)}$ steps where

$$M_{(1.4)} \ge \log_3\left(\frac{\log\eta + \log D_{(1.4)}^{\frac{1}{2}}}{\log r^{(0)} + \log D_{(1.4.)}^{\frac{1}{2}}}\right) \tag{3.2}$$

with the definition that

$$D_{(1.4)} = \frac{2(n-1)}{(\rho^{(0)} - \frac{7}{3} r^{(0)})^2} \tag{3.3}$$

The proof of (3.2) is made available from the fact that for the iterative method of order p=3, $r^{(s)} = D_{(1.4)}(r^{(s)})^p$. Thus

$$r^{(s)} = D_{(1.4)}^{\left(\frac{1-p^s}{1-p}\right)}(r^{(0)})^{p^s}\ ;\ p = 3 \tag{3.4}$$

in (2.9'). Equivalently

$$D_{(1.4)}^{\left(\frac{1}{1-p}\right)} r^{(s+M)} = \left(D_{(1.4)}^{\left(\frac{1}{1-p}\right)} r^{(s)} \right)^{p^M} \quad ; \; p = 3 \tag{3.5}$$

Derive the relation (3.2) from this by setting s=0 and $r^{(M)} = \eta$ and obtaining $M_{(1.4)}$. It is worth remarking that convergence can be inferred from having

$$r^{(0)} < \frac{1}{D_{(1.4)}^{\left(\frac{1}{1-p}\right)}} = \frac{1}{\left(\dfrac{2(n-1)}{(\rho^{(0)} - \frac{7}{3} r^{(0)})^2} \right)^{\frac{1}{2}}} \; ; \; p = 3 \tag{3.6}$$

One wonders how computationally useful this implicit requirement may be? The similar result of (3.2) is:

$$M \geq \log_b \left(\frac{\log \eta + \log D^{\frac{1}{p}}}{\log r^{(0)} + \log D^{\frac{1}{p}}} \right) \tag{3.7}$$

for Newton's

$$p = 3 \; ; D = D_N = \frac{n-1}{\rho^{(0)} (\rho^{(0)} - 4(n-1) r^{(0)})} \; ; \; b = 3 \tag{3.8}$$

and for Laguerre's

$$p = 4 \; ; D = D_L = \frac{3(n-1)}{(\rho^{(0)} - 2 r^{(0)})^3} \; ; \; b = 4 \tag{3.9}$$

Conclusively, for the tolerance $\eta = 10^{-m} ; m >> 1$ and such that the logarithms in bracket in (3.2), (3.7) are computed in base ten for the same initial disks $Z_i^{(0)}$ then for the methods, m correct significant digits can be attained in $O(\log_3 m)$ steps for the method (1.4) and Newton's method, while it will require an $O(\log_4 m)$ steps for Laguerre's method. The order of steps appear to be bigger for the case of Laguerre's method. This is attributed in part to the fact that the method requires the root of a disk which computation may be ambiguous.

Theorem 3.2

Let $r^{(0)} \leq \frac{\rho^{(0)}}{6(n-1)}$ for which the initial error in the starting disks is such that $\eta \leq r^{(0)}$

then

$$M \geq \left\{ \begin{array}{ll} \frac{\log(\frac{r^{(0)}}{\eta})}{\log 15} & ;\ \textit{Newton's Method} \\ \\ \frac{\log(\frac{r^{(0)}}{\eta})}{\log 130} & ;\ \textit{Laguerre's Method} \\ \\ \frac{\log(\frac{r^{(0)}}{\eta})}{\log 4} & ;\ \textit{Method (1.4)} \end{array} \right\} \tag{3.10}$$

For a proof of the first and second cases see Gargantini [3]. The third is seen from

$$r^{(1)} = \max_{J} \left\{ r_j^{(1)} \right\} < \frac{1}{4} r^{(0)};\ j = 1(1)n \tag{3.11}$$

in (2.21). It is then that

$$r^{(s)} = \left(\frac{1}{4} \right)^{s} r^{(0)} \tag{3.12}$$

and the third relation in (3.10) follows. The comparison of these methods continues in what follows.

4. COMPUTATIONAL COMPLEXITY

Consider now the computational complexities of the methods. In this regard let

$$Z = \{z; r\};\ Z_1 = \{z_1 ; r_1\}; Z_2 = \{z_2 ; r_2\}$$

The relevant disks operations to realize (1.4) are listed as follows:

(1) $c + Z = \{c + z; r\}$; $c \in C$

(2) $cZ = \{cz ;| c | r\}$

(3) $Z_1 + Z_2 = \{z_1 + z_2; r_1 + r_2\}$

(4) $Z^{-1} = \left\{ \frac{\bar{z}}{| z |^2 - r^2}; \frac{r}{| z |^2 - r^2} \right\}$; $0 \notin Z$

where z is the conjugate of $\bar{z}$. The computational complexity of these disks operations are shown in table (3.1). These complexities are optimized in the sense of Gargantini [3]. The method of Newton is given by

$$Z_j^{(s-1)} = z_j^{(s)} - \frac{1}{\frac{1}{F(z_j^{(s)})} - \sum_{i=1:j\neq i}^{n} \frac{1}{z_j^{(s)} - Z_i^{(s)}}} \tag{4.1}$$

and that of Laguerre is

$$Z_j^{(s-1)} = z_j^{(s)} - \frac{1}{\left(H_j^{(s)} - \sum_{i=1:j\neq i}^{n} \left(\frac{1}{z_j^{(s)} - Z_i^{(s)}} \right)^2 \right)^{\frac{1}{2}}} \tag{4.2}$$

with

$$H_i^{(s)} = \left(\frac{P_n(z_i^{(s)})}{P_n(z_i^{(s)})} \right)^2 - \frac{P_n(z_i^{(s)})}{P_n(z_i^{(s)})} \tag{4.3}$$

These methods iterate disks directly. In particular, consider (1.4) in the beginning iteration

$$Z_i^{(1)} = z_i^{(0)} - F(z_i^{(0)}) \left(1 - \sum_{i=1:j\neq i}^{n} \left(\frac{F(z_i^{(0)})}{z_i^{(0)} - Z_i^{(0)}} \right) \right) \tag{4.4}$$

The number of real arithmetic operations to compute $F(z_j^{(0)})$ is in table (3.2), (cf: Gargantini [1])

Table 3.2 Computational Complexity of $F(z_j^{(0)})$

Number of real	$F(z_j^{(0)})$
Additions	14n-1
Multiplications	6n+2
Divisions	2

The total number of real arithmetic operations to compute a component of $\left\{Z_1^{(1)}, Z_2^{(1)}, \ldots, Z_{n-1}^{\prime}, Z_n^{(1)}\right\}$ is given in Table (3.3).

Table 3.3 Computational Complexity of the Methods

	Gargantini [1]		
Number of real:	Newton	Laguerre	Method (1.4)
Additions	21n-5	33n-10	27n-7
Multiplication and Divisions	12n+4	20n+9	18n-2
Elementary functions evaluation/Square root		n+5	n+2
Total	N_T=33n-1	L_T=54n+4	M_T=46n-7

The rating here, with respect to the number of arithmetic operations is

$$0 < N_T < M_T \leq L_T \qquad (4.5)$$

This implies that the Laguerre's methods incures the highest computational complexities per iteration step compared to the other methods. This is due to the need for the squaring of the disks and the subsequent computation of square roots of disks in the denominator of the method. The method (1.4) presents an advantage over Laguerre's method in this regard. The Newton's method appear superior here.

5. EFFICIENCY MEASUREMENT

Finally, another performance measuring parameter for iterative methods is the efficiency index E given by

$$E = \frac{log_2 p}{M} \; ; \; M \neq 0 \tag{5.1}$$

where p is the order of the method and M is the total number of multiplications and divisions per iteration step. It is logical that the smaller the E the more inefficient the method and the larger the E the more efficient the algorithm. If on the average we estimate six multiplications or divisions for the computation of square root, elementary functions then the computational complexities of Newton's is estimated at 12n multiplication/divisions, that of Laguerre's is 26n and that of (1.4) is 24n. Therefore

$$E = \left\{ \begin{array}{ll} \frac{0.13}{n} & ; \; Newton's \;\; Method \\ & \\ \frac{0.077}{n} & ; \; Laguerre's \;\; Method \\ & \\ \frac{0.066}{n} & ; \; Method \; (1.4) \end{array} \right\} \tag{5.2}$$

Again, the Laguerre's method appear to be the most inefficient of these algorithms, while the Newton's algorithm is most efficient of all. In an asymptotic sense of (2.9') the computational work required to attain a prescribed tolerance η will be far more than that of (1.4), again the Newton's algorithm is superior. In a parallel machine of a K number of processor capability, Gargantini [1] has conjectured that for $K \geq 2n$ the parallel Laguerre's method becomes superior from a computational stand-point to the parallel Newton's algorithm. However, the validity of this lies on the implementation of these methods on an actual machine of parallel architectural design.

6. NUMERICAL EXPERIMENT

To show the convergence of algorithm (1.4) consider the following polynomials ; the method (1.4) was applied for the computation of the eigenvalues of the Hessenberg's matrix Petkovic [7, p. 51];

$$H=\begin{bmatrix} 8+12i & 1 & 0 & 0 \\ 0 & 6+9i & 1 & 0 \\ 0 & 0 & 4+6i & 1 \\ 1 & 0 & 0 & 2+3i \end{bmatrix}$$

for which

PI: Det **(A - λ I)** =

$$\lambda^4-(20-30\,\mathrm{i})\lambda^3+(-175+420\,\mathrm{i})\lambda^2+(2300-450\,\mathrm{i})\lambda\ \ -\mathbf{2857-2880\,i}$$

starting initially with the Gerschgorin's disks

$$Z_1^{(0)}=\{8+12i;R^{(0)}\},\quad Z_2^{(0)}=\{6+9i;R^{(0)}\},$$

$$Z_3^{(0)}=\{4+6i;R^{(0)}\},\quad Z_4^{(0)}=\{2+3i;R^{(0)}\},$$

$$R^{(0)}=0.4,0.6,0.8,1.0,1.2,1.4,1.6,1.8,2.0,\quad r^{(s)}=\max_{i\le j\le n}\{r_i^{(s)}\}$$

The results are in table (6.1).

etkovic [7, p. 51]

he next problem considered is the calculation of the roots of the **polynomial** equation; Petkovic [7,p.94]

PII: $z^5 + (-4+i)z^4 + (6-4i)z^3 + (-4+6i)z^2 - (15+4i)z - 15i = 0$

The exact roots and starting disk of equal radius, are

$$\varepsilon_1 = -1,\ \varepsilon_{2,3} = 1 \pm 2i,\ \varepsilon_4 = 3,\ \varepsilon_5 = -i$$
$$z_1^{(0)} = -0.7 + 0.5i,\ z_2^{(0)} = 1.3 - 2.3i,\ z_3^{(0)} = 1.4 + 2.4i,$$
$$z_4^{(0)} = 2.6 + 0.4i,\ z_5^{(0)} = 0.2 - 1.3i,\ r_j^{(0)} = 0.8,\ j = 1(1)5$$

The roots from computations are found in table (6.3).

Table (6.3) : Results of Algorithm (1.4) on PII.

$$Z_1^{(7)} = \{-0.99999999857175 - 0.0000000371269i\ ;\ 0.792565107563\ (-23)\}$$
$$Z_2^{(7)} = \{0.99999999999666 + 2.00000000003234i\ ;\ 0.00665054604878\ (-23)\}$$
$$Z_3^{(7)} = \{1.00000000000093 - 1.99999999999926i\ ;\ 0.00010861564408\ (-23)\}$$
$$Z_4^{(7)} = \{2.99999999999980 - 0.00000000000026i\ ;\ 0.00002974135506\ (-23)\}$$
$$Z_5^{(7)} = \{0.00000003860108 - 0.99999997399240i\ ;\ 0.53063243443032\ (-23)\}$$

The next problem is ; Petkovic and Carstensen [8]

PIII : $P_9(z) = z^9 + 8z^8 - z^7 + 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300 = 0$

$$z_1^{(0)} = \{-3.2 + 0.2i\ ;0.35\},\ z_2^{(0)} = \{-1.1 - 0.2i\ ;0.35\},\ z_3^{(0)} = \{1.7i\ ;0.35\},$$
$$z_4^{(0)} = \{-1.9 + 1.3i\ ;\ 0.35\},\ z_5^{(0)} = \{-1.8 - 0.8i\ ;\ 0.35\},\ z_6^{(0)} = \{2.3 - 1.1i\ ;0.35\},$$
$$z_7^{(0)} = \{2.9 + 1.9i\ ;0.35\},\ z_2^{(0)} = \{-1.1 - 0.2i\ ;0.35\},\ z_3^{(0)} = \{1.7i\ ;0.35\},$$
$$z_4^{(0)} = \{-1.9 + 1.3i\ ;\ 0.35\},\ z_9^{(0)} = \{0.2 - 2.2i\ ;0.35\},$$

The roots are -3, -1, 2i, -2 ±i, 2± i , 1, -2i and the numerical results after seventh - iteration are in table (6.4) , although with some slightly different starting iterates.

Table (6.4) : Results of Algorithm (1.4) on PIII.

$Z_1^{(7)}$ = {-2.9999999933049 - 0.00000000563166i; 0.00244675715176 (-19)}

$Z_2^{(7)}$ = {-1.00000000352870-0.00000001404681i ; 0.00883916034499 (-19)}

$Z_3^{(7)}$ = {0.00000006302689 + 1.99999998789414i ; 0.01664763542100 (-19)}

$Z_4^{(7)}$ = {-2.00000000083152 + 0.99999999934021i ; 0.00027472017235 (-19)}

$Z_5^{(7)}$ = {-1.99998881561758 - 0.99999999929793i ; 0.170237713236 71(-19)}

$Z_6^{(7)}$ = {2.00000000026917- 0.99999999929793i ; 0.00014203749817 (-19)}

$Z_7^{(7)}$ = {1.99998202238249 +0.99998141223742i ; 0.18966744840734 (-19) }

$Z_8^{(7)}$ = {1.00000000202584 + 0.00000000455847i ; 0.00211408239406 (-19)}

$Z_9^{(7)}$ = {-0.00000000185156 - 2.00000000057268i ; 0.00037414193826 (-19)}

From the numerical examples, it is deductive that algorithm (1.4) converges, in fact, these results compare with the references cited, although the radii of disks obtained from computation over estimates the error bounds on the zeros. This may be corrected by adopting a disk inversion with bigger radius. In this regard therefore the disks inversion

$$\{z;r\}^{I_\nu} = \left\{ \begin{array}{ll} \left\{ \frac{1}{z}; \frac{r}{|z|(|z|-r)} \right\} & ;\nu = 1 \\ \\ \left\{ \frac{1}{z}; \frac{2r}{|z|^2 - r^2} \right\} & ;\nu = 2 \end{array} \right\}$$

defined in Petkovic and Carstensen [8] in place of that in (4) in section (4) may be found useful.

7. CONCLUSION

Conclusively, the need to compute the zeros of polynomials arises naturally in the stability analysis of initial value solvers in ordinary differential equations and more so in other areas of practical applications. We have presented a new interval method in (1.4) along with its convergence analysis. The performance, efficiency and computational complexity compares with results from Petkovic [7] and Gargantini [3].

In the implementation of (1.4) the definition $F(z_j) = \frac{P_n(z_j)}{P_n'(z_j)}$ was employed, although the alternative definition $F(z_j) = \frac{P_n(z_j)}{\prod_{i=1, j \neq 1}^{n} \left(z_j - z_i\right)}$ yields improved results. However, we remark that because of the inclusion relation

$$\sum_{\substack{j=1 \\ i \neq j}}^{n} \left(\frac{F\left(z_j^{(s)}\right)}{z_i^{(s)} - Z_j^{(s)}} \right) \subset \left\{ \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{F\left(z_j^{(s)}\right)}{z_i^{(s)} - z_j^{(s)}}; \frac{r^{(s)}}{\left(\rho^{(s)}\right)^2} \sum_{\substack{i=1 \\ i \neq j}}^{n} \left|F\left(z_i^{(s)}\right)\right| \right\} \subset \left\{ \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{F\left(z_j^{(s)}\right)}{z_i^{(s)} - z_j^{(s)}}; \frac{(n-1) r^{(s)}}{\left(\rho^{(s)}\right)^2} \underset{\substack{i=1(1)n \\ i \neq j}}{\mathrm{Max}} \left|F\left(z_i^{(s)}\right)\right| \right\}$$

obtained by appealing to (1.5a), the interval method (1.4) is faster in convergence than its new variant (1.5)

$$W_i^{(s)} = z_i^{(s)} - F\left(z_i^{(s)}\right) \left(1 - \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{F\left(z_j^{(s)}\right)}{z_i^{(s)} - z_j^{(s)}}; \frac{r^{(s)} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|F\left(z_j^{(s)}\right)\right|}{\left(\rho^{(s)}\right)^2} \right\} \right) \tag{7.1}$$

$$Z_j^{(s+1)} = W_j^{(s)} \cap Z_j^{(s)}; j = 1(1)n; s = 0,1,2,\ldots$$

considered in Ikhile [10].

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