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SOLUTION OF LARGE SYSTEMS OF LINEAR EQUATIONS USING THE METHOD OF ALTERNATING DIRECTION IMPLICIT (ADI) AS PRECONDITIONER

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ABSTRACT: This paper solves the discretized 3 – dimensional Poisson equation. The method of Alternating Direction Implicit (ADI) is applied as a preconditioner with the Conjugate Gradient Method (CGM). A fast convergence to the solution is attained.

1. INTRODUCTION:

In optimization studies, one of the current research interest is to solve large system of equations. Many problems in practice require the solution of very large systems of linear equations. The wide range of areas includes econometrics, stress analysis, metreology, fluid flow problem, semiconductors and aerodynamics. Also, the current developments in space technology tend toward the deployment of ever-larger structures in space. Partial differential equations when discretised using finite element or finite difference method often gives rise to large scale systems. The characteristics of the resultant discretised systems and the large systems of equations are that they either sparse or dense. Fortunately, many of these large systems of linear equations $Ax = b$ have A as sparse i.e. relatively few nonvanishing elements. The solution of these linear systems can be accomplished by direct or iterative methods.

Direct methods can lead to high computational complexity and to considerable memory requirements both of which limit the practical size of the discretization that can be used to solve the system. These memory requirements arise from filling in to form the matrix factorisation and from pivoting which may be required in solving other than positive definite symmetric problems. As an alternative approach iterative methods have been considered which permit the maintenance of the sparsity pattern, reducing storage requirements while decreasing computation time as well. A variety of techniques use a combination of direct and iterative approaches to obtain the advantages of both methods. Miejerink and Van der Vorsk [1], and Kershaw [2] had successfully used iterative methods for the approximate solutions of such problems. Since then, the interest had been sustained. When the size of the matrix A is large, the convergence of the iterative process is slow. It is necessary to accelerate the convergence process. A well known technique to accelerate convergence is known as preconditioning.

2. THE MODEL PROBLEM

Mathematical Formulation

Our model problem is the 3-dimensional Poisson equation of the form

$$\nabla(K\nabla u) = F; \qquad (x, y, z) \in \Omega \tag{2.1}$$

Where K and F are given functions of the three spatial variables.
From (2.1), we have

$$\nabla K.\ \nabla u + K\nabla^2 u = F. \tag{2.2}$$

Assuming K is a constant, then (2.2) becomes

$$\nabla^2 u = \bar{F}; \qquad \bar{F} = F/K. \tag{2.3}$$

If $\bar{F} \equiv 0$, we have the Laplace equation. Let u(x, y, z) represent the equilibrium temperature distribution in a 3-dimensional heat- conducting medium Ω defined on a cube $0 < x, y, z < 1$. To obtain a system of finite difference equation for (2.2),[3] we approximate the derivatives by the central difference schemes to get the following: Where we have assumed equal grid size h on all dimension.

$$\left[\frac{\partial k}{\partial x} + \frac{\partial k}{\partial y} + \frac{\partial k}{\partial z}\right].\left\{\frac{u_{i+1,j,k} - u_{i-1,j,k}}{2h} + \frac{u_{i,j+1,k} - u_{i,j-1,k}}{2h} + \frac{u_{i,j,k+1} - u_{i,j,k-1}}{2h}\right\} + k(x,y,z)\left\{\frac{u_{i-1,j,k} - 2u_{i,j,k} + u_{i+1,j,k}}{h^2} + \frac{u_{i,j-1,k} - 2u_{i,j,k} + u_{i,j+1,k}}{h^2} + \frac{u_{i,j,k-1} - 2u_{i,j,k} + u_{i,j,k+1}}{h^2}\right\}$$
$$= F(x,y,z) \tag{2.4}$$

If K (x, y, z) $\equiv$ xyz and

$$F(x,y,z) \equiv \left(\sin\frac{\pi}{2}x + Cos\frac{\pi}{2}y + z\right),$$

then (2.4) becomes

$$2(xyz).\{u_{i-1,j,k} + u_{i+1,j,k} + u_{i,j-1,k} + u_{i,j+1,k} + u_{i,j,k-1} + u_{i,j,k+1} - 6u_{i,j,k}\}$$
$$+ h(yz + xz + xy)\{u_{i+1,j,k} - u_{i-1,j,k} + u_{i,j+1,k} - u_{i,j-1,k} + u_{i,j,k+1} - u_{i,j,k-1}\}$$

$$= \quad -2h^2\left(\sin\frac{\pi}{2}x + Cos\frac{\pi}{2}y + z\right) \tag{2.5}$$

Defining approximations u_{ijk} to the exact solutions u(x,y,z) at the N^3 interior grid points, (2.5) gives

$$2(ijk)\left\{-u_{i-1,j,k} - u_{i+1,j,k} - u_{i,j-1,k} - u_{i,j+1,k} - u_{i,j,k-1} - u_{i,j,k+1} + 6u_{i,j,k}\right\}$$
$$+h(jk+ik+ij)\left\{-u_{i+1,j,k} - u_{i-1,j,k} - u_{i,j+1,k} - u_{i,j-1,k} - u_{i,j,k+1} - u_{i,j,k-1}\right\}$$

$$= -2h^2\left(\sin\frac{\pi}{2}x + Cos\frac{\pi}{2}y + z\right), \qquad i,j,k = 1,2,\cdots,N. \tag{2.6a}$$

If we assume the Dirichlet boundary conditions;

$$u(0,y,z) = u(x,0,z) = u(x,y,0) = 0$$
$$u(1,y,z) = u(x,1,z) = u(x,y,1) = 0, \tag{2.6b}$$

Equation (2.6) is therefore a system of N^3 equations in the N^3 unknown u_{ijk}, i, j, k = 1,2, ..., N corresponding to the interior grid points, h = 1/N.

Analysis

Equation (2.6) is written in matrix – vector form. For this purpose, the interior grid points are numbered in a form called the natural or row – wise ordering. Corresponding to this ordering of the grid points, we order the unknown U_{ijk} into the vector.

$$(U_{111}, U_{211}, \ldots, U_{N11}; U_{121}, U_{221}, \ldots, U_{N21}; \ldots; U_{NN1}, U_{NN2}, \ldots, U_{NNN}) \tag{2.7}$$

In this wise, writing the system of equation (2.6) in block form, for N = 3, we have the coefficient matrix become:

$$A_{nn} = \begin{bmatrix} B & I & & & I & & & \\ I & B & I & & & I & & \\ & I & B & & & & I & \\ I & & & B & I & & I & \\ & I & & I & B & I & & I \\ & & I & & I & B & & I \\ & & & I & & & B & I \\ & & & & I & & I & B \end{bmatrix} \tag{2.8}$$

Here, the blocks B are square matrices of order n and I denote the n x n identity matrix. Equation (2.6) becomes the N^3 x N^3 coefficient matrix in block tridiagonal matrix (2.8)[4].

3. METHOD OF ALTERNATING DIRECTION (ADI)

Application of the Alternating Direction implicit iteration method to our model problem (2.1), and with the central difference discretization given in (2.7), the matrix A can be splitted into

$$A = B + C + D \tag{3.1}$$

Here, B, C, D are defined through their actions on a vector u:

$$\begin{aligned} W_{ijk} &= 2u_{ijk} - u_{i-1,jk} - u_{i+1,jk}, && \text{if } W = Bu \\ W_{ijk} &= 2u_{ijk} - u_{i,,j-1,,k} - u_{i,j+1,k}, && \text{if } W = Cu \\ W_{ijk} &= 2u_{ijk} - u_{ij,k-1} - u_{i,j,k+1}, && \text{if } W = Du \end{aligned} \tag{3.2}$$

B, C, D are symmetric and positive definite block diagonal matrices. In the ADI method of Peaceman and Rachford [5] the system of equations $Ax = b$, in accordance with the decomposition A = B + C + D, is now transformed equivalently into

$$B + 1/2D + \omega I = \omega I - C - 1/2D \tag{3.3a}$$

and also

$$C + 1/2D + \omega I = \omega I - B - 1/2D \tag{3.3b}$$

where ω is an arbitrary real parameter which is chosen in such a way that the method converges as fast as possible. As a relaxation method, we chose $0 < \omega < 2$[6]. With the abbreviations $B_1 = B + 1/2D$, $C_1 = C + 1/2D$, the first half-step of the ADI method corresponds to the splitting

$$A = W - R \text{ with } W = \omega I + B_1,\ R = \omega I - C_1.$$

Interchangeably the roles of B and C, that is, alternating the direction, we generate the splitting of the second half-step

$$A = W - R \text{ with } W = \omega I + C_1,\ R = \omega I - B_1.$$

Hence, the complete ADI method is the following

$$M_{\omega}^{ADI} = (\omega I + C_1)^{-1}(\omega I - B_1)(\omega I + B_1)^{-1}(\omega I - C_1). \tag{3.4}$$

We have Bu in blocks as

$$Bu = \begin{bmatrix} B &&&&&&&& \\ & B &&&&&&& \\ && B &&&&&& \\ &&& B &&&&& \\ &&&& B &&&& \\ &&&&& B &&& \\ &&&&&& B && \\ &&&&&&& B & \\ &&&&&&&& B \end{bmatrix} \text{ where } B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix} \tag{3.5}$$

also Cu in blocks as

$$Cu = \begin{bmatrix} C & -I &&&&&&& \\ -I & C & -I &&&&&& \\ & -I & C &&&&&& \\ &&& C & -I &&&& \\ &&& -I & C & -I &&& \\ &&&& -I & C &&& \\ &&&&&& C & -I & \\ &&&&&& -I & C & -I \\ &&&&&&& -I & C \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

I is the 3X3 identity matrix

$$Du = \begin{bmatrix} D & & & -I & & & & & \\ & D & & & -I & & & & \\ & & D & & & -I & & & \\ -I & & & D & & & -I & & \\ & -I & & & D & & & -I & \\ & & -I & & & D & & & -I \\ & & & -I & & & D & & \\ & & & & -I & & & D & \\ & & & & & -I & & & D \end{bmatrix}$$

while

$$B_1 = R + 1/2D = \begin{bmatrix} S & & & -\frac{1}{2}I & & & & & \\ & S & & & -\frac{1}{2}I & & & & \\ & & S & & & -\frac{1}{2}I & & & \\ -\frac{1}{2}I & & & S & & & -\frac{1}{2}I & & \\ & -\frac{1}{2}I & & & S & & & -\frac{1}{2}I & \\ & & -\frac{1}{2}I & & & S & & & -\frac{1}{2}I \\ & & & -\frac{1}{2}I & & & S & & \\ & & & & -\frac{1}{2}I & & & S & \\ & & & & & -\frac{1}{2}I & & & S \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$
\omega I + B_1 = \begin{bmatrix}
U & & & -\frac{1}{2}I & & & & & \\
& U & & & -\frac{1}{2}I & & & & \\
& & U & & & -\frac{1}{2}I & & & \\
-\frac{1}{2}I & & & U & & & -\frac{1}{2}I & & \\
& -\frac{1}{2}I & & & U & & & -\frac{1}{2}I & \\
& & -\frac{1}{2}I & & & U & & & -\frac{1}{2}I \\
& & & -\frac{1}{2}I & & & U & & \\
& & & & -\frac{1}{2}I & & & U & \\
& & & & & -\frac{1}{2}I & & & U
\end{bmatrix}
$$

$$
U = \begin{bmatrix}
\frac{7}{2} & -1 & 0 \\
-1 & \frac{7}{2} & -1 \\
0 & -1 & \frac{7}{2}
\end{bmatrix}
$$

$$C_1 = C + 1/2D = \begin{bmatrix} T & -I & & -\frac{1}{2}I & & & & & \\ -I & T & -I & & -\frac{1}{2}I & & & & \\ & -I & T & & & -\frac{1}{2}I & & & \\ -\frac{1}{2}I & & & T & -I & & -\frac{1}{2}I & & \\ & -\frac{1}{2}I & & -I & T & -I & & -\frac{1}{2}I & \\ & & -\frac{1}{2}I & & -I & T & & & -\frac{1}{2}I \\ & & & -\frac{1}{2}I & & & T & -I & \\ & & & & -\frac{1}{2}I & & -I & T & -I \\ & & & & & -\frac{1}{2}I & & -I & T \end{bmatrix}, \quad T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Where we have chosen x = ½.

$$\omega I - B_1 = \begin{bmatrix} V & & & -\frac{1}{2}I & & & & & \\ & V & & & -\frac{1}{2}I & & & & \\ & & V & & & -\frac{1}{2}I & & & \\ -\frac{1}{2}I & & & V & & & -\frac{1}{2}I & & \\ & -\frac{1}{2}I & & & V & & & -\frac{1}{2}I & \\ & & -\frac{1}{2}I & & & V & & & -\frac{1}{2}I \\ & & & -\frac{1}{2}I & & & V & & \\ & & & & -\frac{1}{2}I & & & V & \\ & & & & & -\frac{1}{2}I & & & V \end{bmatrix} \quad V = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 1 & -\frac{3}{2} & 1 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

$$
\omega I + C_1 = \begin{bmatrix}
F & -I & & -\frac{1}{2}I & & & & & \\
-I & F & -I & & -\frac{1}{2}I & & & & \\
& -I & F & & & -\frac{1}{2}I & & & \\
-\frac{1}{2}I & & & F & -I & & -\frac{1}{2}I & & \\
& -\frac{1}{2}I & & -I & F & -I & & -\frac{1}{2}I & \\
& & -\frac{1}{2}I & & -I & F & & & -\frac{1}{2}I \\
& & & -\frac{1}{2}I & & & F & -I & \\
& & & & -\frac{1}{2}I & & -I & F & -I \\
& & & & & -\frac{1}{2}I & & -I & F
\end{bmatrix}, \quad F = \begin{bmatrix} \frac{7}{2} & 0 & 0 \\ 0 & \frac{7}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}
$$

$$
\omega I - C_1 = \begin{bmatrix}
G & I & & \frac{1}{2}I & & & & & \\
I & G & I & & \frac{1}{2}I & & & & \\
& I & G & & & \frac{1}{2}I & & & \\
\frac{1}{2}I & & & G & I & & \frac{1}{2}I & & \\
& \frac{1}{2}I & & I & G & I & & \frac{1}{2}I & \\
& & \frac{1}{2}I & & I & G & & & \frac{1}{2}I \\
& & & \frac{1}{2}I & & & G & I & \\
& & & & \frac{1}{2}I & & I & G & I \\
& & & & & \frac{1}{2}I & & I & G
\end{bmatrix} \qquad G = \begin{bmatrix} -\frac{3}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix},
$$

$$
(\omega I + B_1)^{-1} = \begin{bmatrix}
U^{-1} & & & -2I & & & & & \\
& U^{-1} & & & -2I & & & & \\
& & U^{-1} & & & -2I & & & \\
-2I & & & U^{-1} & & & -2I & & \\
& -2I & & & U^{-1} & & & -2I & \\
& & -2I & & & U^{-1} & & & -2I \\
& & & -2I & & & U^{-1} & & \\
& & & & -2I & & & U^{-1} & \\
& & & & & -2I & & & U^{-1}
\end{bmatrix}, \quad U^{-1} = \frac{1}{287}\begin{bmatrix} 90 & 28 & 8 \\ 28 & 98 & 28 \\ 8 & 28 & 90 \end{bmatrix}
$$

$$
(\omega I + C_1)^{-1} = \begin{bmatrix}
F^{-1} & & & -2I & & & & & \\
& F^{-1} & & & -2I & & & & \\
& & F^{-1} & & & -2I & & & \\
-2I & & & F^{-1} & & & -2I & & \\
& -2I & & & F^{-1} & & & -2I & \\
& & -2I & & & F^{-1} & & & -2I \\
& & & -2I & & & F^{-1} & & \\
& & & & -2I & & & F^{-1} & \\
& & & & & -2I & & & F^{-1}
\end{bmatrix}, \quad F^{-1} = \begin{bmatrix} \frac{2}{7} & 0 & 0 \\ 0 & \frac{2}{7} & 0 \\ 0 & 0 & \frac{2}{7} \end{bmatrix}.
$$

From which we have equation (3.4) becoming

4. COMPUTATIONAL RESULTS

Table 3.1: Number Of Iterations For The Convergence Of The 3-Dimentional Poisson Equation (2.1) Using The Method Of Alternating Direction (ADI)

	NUMBER OF ITERATIONS	
	W = ½	W = 9/5
N = 3: 27 X 27 MATRIX	1	1
N = 4: 64 X 64 MATRIX	1	2
N = 5: 125 X 125 MATRIX	2	2
N = 6: 216 X 216 MATRIX	2	2
N = 8: 512 X 512 MATRIX	2	2
etc.		

5. CONCLUDING REMARKS

From the convergence result presented, it is interesting to observe that the preconditioning technique worked very well with the computational exposition carried out. The condition number ranged from 1.5 to 1.8 for all sizes of matrices considered.

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