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ON SOME PROBLEMS OF THE VARIABLE END-POINT THEOREM

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ABSTRACT

In determining a minimizing arc in a class of admissible arcs joining an end curve to a fixed point or an end curve to another end curve, the variable end-point theory often proves to be a very efficient mathematical tool. This theorem is utilized in this study to analyse the external curve joining

(a) a hyperbolic end curve and a specified origin

(b) a hyperbolic end curve and a circular end curve.

(a) was proposed by Glegg (1968), whilst (b) is the generalization. Further, (b) usually involves a large number of unknowns. This study intends to suggest an efficient analytical method of solution and also; in cases where the geometry of the end curves may be more complex.

The analyses is then applied to study the mass transport across a model river flowing through a channel. Consequently, in section (2) and fig. 1, it is deduced that the mass flow velocity of 11.2msec^{-1} is sufficient to move a considerable load of sedimental layers across the river. In this calculation, only the length scales representing the width of the model river and the focal distance of the bounding hyperbola from a specified origin are involved. In fig. II of section (3), however, a more interesting analysis arises as a result of the inclusion of the radius R of the other end bounding circular curve.

1. INTRODUCTION

Euler's equation which is a necessary condition for weak relative minimum of an integral $\int_a^b F(x,y,y')\,dx$ is as follows

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x,y,y')\right)=\frac{\partial F}{\partial y}(x,y,y'),\;\; y'=\frac{dy}{dx};$$

which modifies to

$$\frac{d}{dx}\left(F-y'\frac{\partial F}{\partial y'}\right)=0,\;\; if\;\; F=F\left(y,y'\right)$$

Thus, $F-y'\,{\partial F}/{\partial y'}=A,\; A = are\;\; cons\tan t,.$

is called the integrated form of Euler's equation.

For detail account of these extensively studied variational analyses, one may mention the following; Glegg (1968) and Pars (1962).

Further but briefly, the variable end-point theorem involves the determination of a minimizing arc $\Gamma_0 : y=\phi(x)$ for the integral (functional) of the form

$$I(\Gamma) = \int_{x_1}^{x_2} F(x, y, y')\,dx \qquad (1.1)$$

in a class of arcs $\Gamma_\alpha : Y = \phi(x) + \alpha\,\eta(x)$ joining two end curves. Where $x_1 = x_1(\alpha), x_2 = x_2(\alpha), y_1 = y(x_1)$ *and* $y_2 = y(x_2)$ in the xy space, α being a parameter for the end-curves. As the two end curves are varying, a complex form of equations arise. The aim of this study is to suggest an extended and simplified method of solving the equations so derived.

2. A SIMPLE APPLICATION OF VARIABLE END-POINT THEOREM.

As proposed by Glegg (1968), consider the minimzation of the integral

$$\int_0^1 y^2 y'^2\, dx,$$

where the end-points are (0,0) and (x_1, y_1), the later end-point lies on the hyperbola $y^2 - x^2 = c^2$. c being a given constant.

This can be analysed usually as follows:

Since $F = F(y, y') = y^2 y'^2$ we now use the integrated form of Euler's equation

$$F - y'\frac{\partial F}{\partial y'} = A$$

$$y^2 y'^2 - 2y'^2 y^2 = A$$

$A = -y'^2 y^2$, A being the same constant of interpration.

Taking square roots of both sides yields

$$A_o = y'y, \quad where\ A_o = \sqrt{-A}\ , A < 0$$

$$A_o = \frac{dy}{dx}y, \quad from\ which\ A_o dx = ydy.$$

Integrating both sides yields

$A_o x + k = \frac{y^2}{2}$, from which

$$y^2 = A_1 x + k_1 \qquad (2.1a)$$

$where\ A_1 = 2A_o, 2k = k_1$

The extremal passing through the end points (0,0) implies that

$A_1 0 + k_1 = 0$, $k_1 = 0$; consequently,

$$y^2 = A_1 x \qquad (2.1b)$$

If the extremal (2.1b) meets the other end point at (x_1, y_1),

We have the following:

$$y_1^2 = A_1 x_1 \qquad (2.2)$$

$$y_1^2 - x_1^2 = c^2 \qquad (2.3)$$

Using variable end-point conditions at (x_1, y_1) as stated by Glegg (1968), Pars (1962) we have:

$$U_1' dx_1 + V_1 dy_1 = 0 \qquad (2.4)$$

Where dx_1 and dy_1 are elements of the curve $y^2 - x^2 = c^2$

$$V = \frac{\partial F}{\partial y'} = 2y^2 y' \quad and \quad U = F - y' \frac{\partial F}{\partial y'} = -y'^2 y^2$$

substituting for V and U in (2.4) yields

$$-y_1'^2 y_1^3 dx_1 + 2y^2 y_1' dy_1 = 0$$

dividing through by $-y_1' y^2$ we have

$$-y_1' \ dx_1 + 2dy_1 = 0 \qquad (2.5)$$

where y_1' is the gradient of the extremal measured along the end curve ($y^2 - x^2 = c^2$).
Differentiating (2.2) implicitly yields

$$2y_1 y_1' = A_1$$

$$\therefore \ y_1' = \frac{A_1}{2y_1} \qquad (2.6)$$

substituting (2.6) in (2.5) yields

$$\frac{-A_1}{2y_1} dx_1 + 2dy_1 = 0$$

$$dy_1 = \frac{A_1 \, dx_1}{4y_1}$$

Differentiating (2.3) implicitly yields

$$2y_1 \, dy_1 = 2x_1 . dx_1$$

$$dy_1 = \frac{x_1}{y_1} dx_1$$

$$\frac{x_1}{y_1} dx_1 = \frac{A_1}{4y_1} dx_1$$

$$A_1 = 4x_1 \qquad (2.7)$$

Substituting the value of A_1 in (2.3) yields

$$y_1^2 = 4x_1^2$$

substituting $y_1^2 = 4x_1^2$ in (2.3) yields

$$3x_1^2 = c^2$$

$$x_1 = \pm \frac{c}{\sqrt{3}}$$

but $y_1 = \sqrt{4x_1^2} = \pm 2x_1 = \pm\frac{2c}{\sqrt{3}}$.

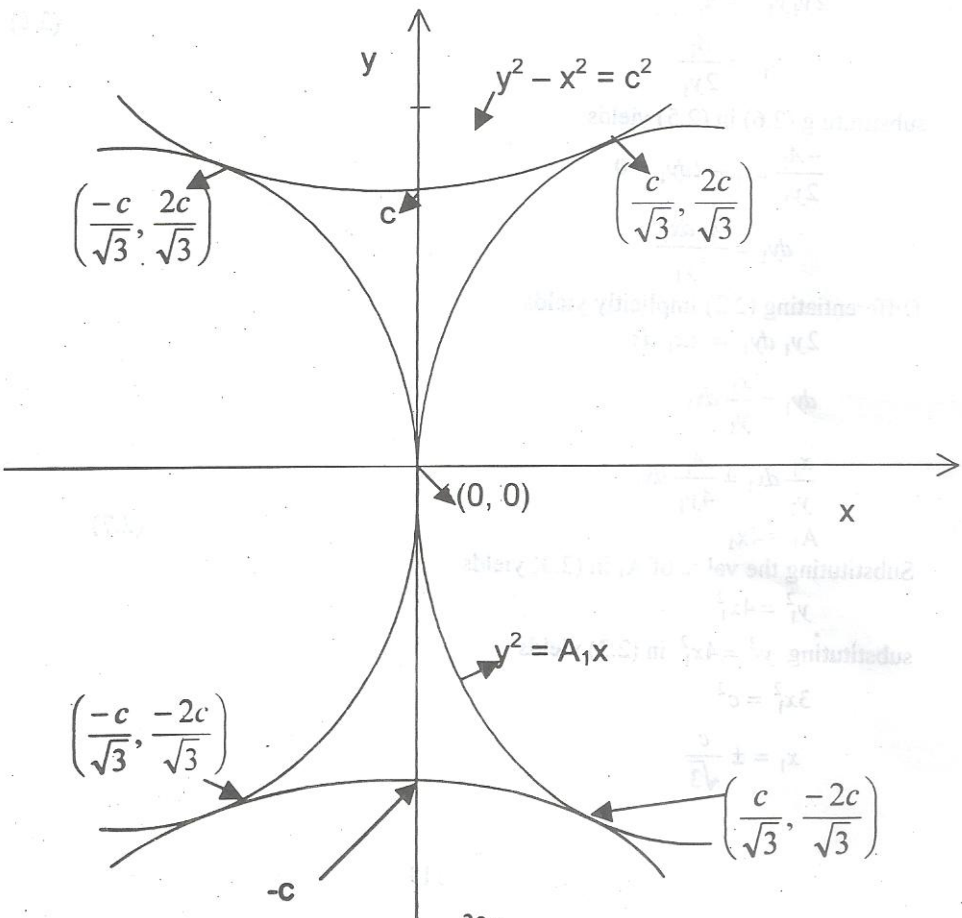
Thus (2.7) is now $A_1 = \pm\frac{4c}{\sqrt{3}}$

and the required extremal is

$$y^2 = \frac{4cx}{\sqrt{3}}; \qquad (2.8)$$

Equation (2.8) is a parabola and c as the factor which is related to its focal distance from a specified origin.

Fig. 1: Diagramatic representation of the extremal passing through origin and the end point (x_1, y_1) lying on the hyperbola.



3 A GENERALIZED FORM OF APPLICATION OF THE ABOVE

Now, considering the minimization of $\int_{x_1}^{x_2} y^{12} y^2\, dx$ with an end points (x_1, y_1) on the hyperbola $y^2 - x^2 = c^2$ and now, the other end point (x_2, y_2) lies on the circle $x^2 + y^2 = R^2$ where c and R are given constants.

As in the previous case, we have from (2.1a)

$y^2 = A_1x + k$.

Through (x_1, y_1) we have the following:

$$y_1^2 = A_1 x_1 + k_1 \quad (3.1)$$

$$y_1^2 - x_1^2 = c^2 \quad (3.2)$$

Also through (x_2, y_2) we have the following

$$y_2^2 = A_1 x_2 + k_1 \quad (3.3)$$

$$x_2^2 + y_2^2 = R^2 \quad (3.4)$$

Using variable end point conditions at (x_1, y_1) we have

$U_1dx_1 + V_1dy_1 = 0$, where dx_1 and dy_1 are the elements of the curve (3.2)

$y^2 - x^2 = c^2$; and (2.5) gives

$$-y_1' \; dx_1 + 2dy_1 = 0 \quad (3.5)$$

where y_1' is the gradient of the extremal measured along the end curves $(y^2 - x^2 = c^2)$ and y_2' is the gradient of the extremal measured along the circle $x^2 + y^2 = R^2$.

Differentiating (3.1) implicitly yields

$$2\, y_1 y_1' = A_1$$

which yields (2.6)

Also using (2.5) and (2.6) we have

$$\frac{-A_1}{2y_1} dx_1 + 2dy_1 = 0, \quad dy_1 = \frac{A_1}{4y_1} dx_1 \; .$$

But from (3.2) i.e differentiating implicitly yields $dy_1 = \frac{x_1}{y_1} dx_1$

$$\frac{x_1}{y_1} = dx_1 = \frac{A_1\, dx_1}{4y_1}$$

$A_1 = 4x_1$, which is (3.7)

Substituting (2.7) in (3.1) yields

$$y_1^2 = 4x_1^2 + k_1 \quad (3.6)$$

Substituting (3.6) in (3.2) yields

$$4x_1^2 + k_1 - x_1^2 = c^2$$

$$3x_1^2 + k = c^2 \quad (3.7)$$

Also differentiating (3.3) implicitly yields

$$2y_2\, y_2' = A_1$$

$$y_2' = \frac{A_1}{2y_2} \tag{3.8}$$

Using (3.5) and (3.8) yields

$$\frac{-A_1}{2y_2}\, dx_2 + 2dy_2 = 0$$

$$dy_2 = \frac{A_1}{4y_2}\, dx_2$$

But differentiating (3.4) implicitly we have,

$$2x_2\, dx_2 + 2y_2\, dy_2 = 0, \quad dy_2 = \frac{-x_2\, dx_2}{y_2}$$

$$\frac{-x_2\, dx_2}{y_2} = \frac{A_1\, dx_2}{4y_2}$$

$$A_1 = -4x_2 \tag{3.9}$$

from (2.7) and (3.9)

substituting (3.9) in (3.3) yields

$$y_2^2 = -4x_2^2 + k_1 \tag{3.10}$$

substituting (3.10) in (3.4) yields

$$-4x_2^2 + k_1 + x_2^2 = R^2$$

$$-3x_2^2 + k_1 = R^2 \tag{3.11}$$

Solving equation (3.7) and (3. 11) simultaneously we have

$$3x_1^2 + k_1 = c^2$$

$$-3x_2^2 + k_1 = R^2$$

$$x_1 = -x_2 \Rightarrow x_1^2 = x_2^2$$

since $k_1 = \dfrac{c^2 + R^2}{2} > 0$

substituting the value of k_1 in (3.7) yields

$$3x_1^2 + \frac{c^2 + R^2}{2} = c^2$$

$$3x_1^2 = \frac{c^2 - R^2}{2}$$

$$x_1 = \pm\sqrt{\frac{c^2 - R^2}{6}}$$

$$x_2 = \mp\sqrt{\frac{c^2 - R^2}{6}}$$

substituting the value of x_1 in (3.2) yields

$$y_1^2 = \frac{c^2 - R^2}{6} + c^2$$

$$y_1 = \pm\sqrt{\frac{7c^2 - R^2}{6}} \qquad 7c > R$$

substituting the value of x_2 in (3.4) yields

$$y_2^2 = R^2 - \left(\frac{c^2 - R^2}{6}\right)$$

$$y_2^2 = \pm\sqrt{\frac{7R^2 - c^2}{6}} \qquad 7R > c$$

we know that $A_1 = 4x_1$, $\therefore\ A_1 = 4\sqrt{\frac{c^2 - R^2}{6}} > 0$

substituting the value of A_1 in (2.1a) yields

$$y^2 = 4x\sqrt{\frac{c^2 - R^2}{6}} + \frac{R^2 + c^2}{2}$$

which is the required extremal.

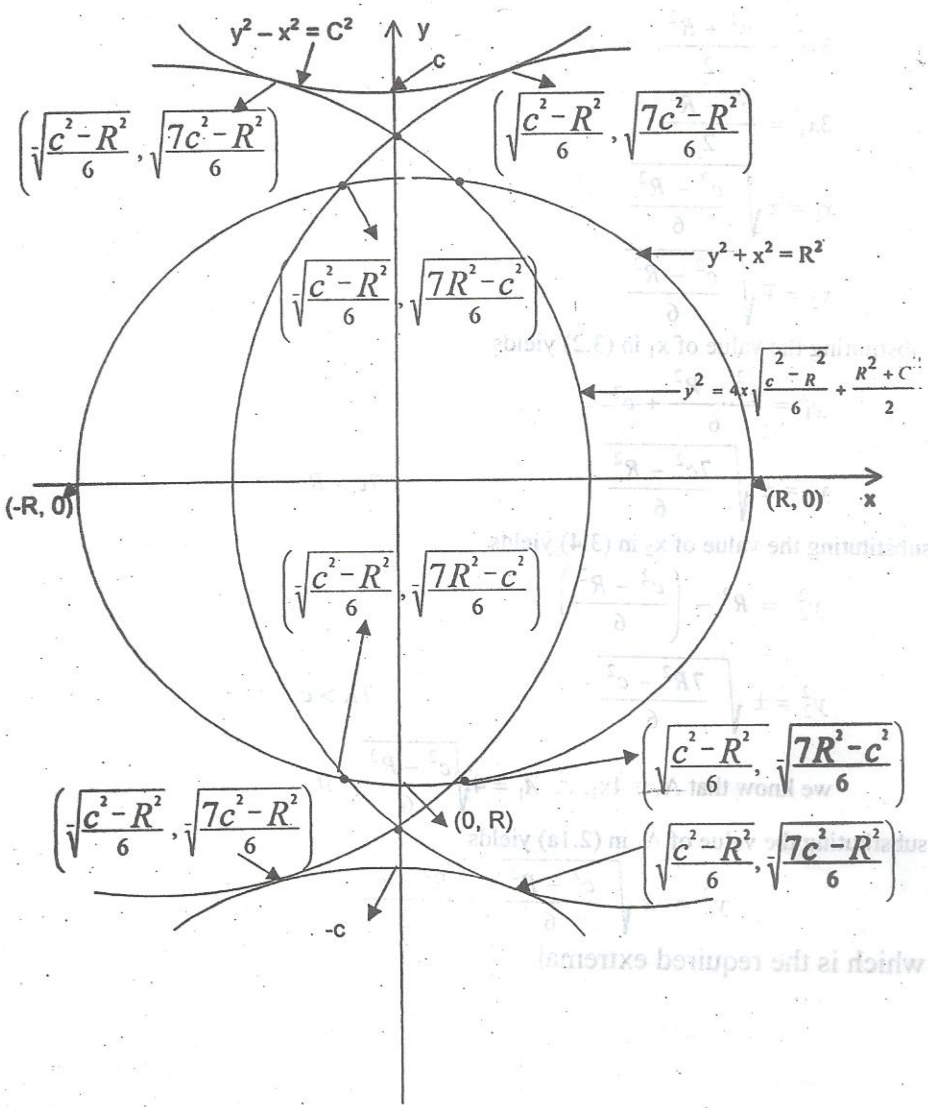


Fig. 2: Diagramatic representation of the extremal with end point (x_1, y_1) lying on the hyperbola and end point (x_2, y_2) lying on the circle.

3. DISCUSSION

The analysis considered in this study may explain a variety of physical and engineering processes. This, however, depends on the interpretations assigned to the variables of the Lagrange's density $F(x, y') = x^2 y'^2$. In an identical consideration, Okeke (1985) applied the generalized Du Bois-Raymond equation to explain successfully, the processes leading to the breaking of the linear waves in the neighbourhood of a shoreline.

However, our interest is essentially on the channel flow and the related mass transport in rivers and estuaries. In this consideration, we take x – axis as horizontal and y – axis as perpendiculars which is also directed downwards. $y' = dy/dx = c_0(x)/\sqrt{gh}$; $c_0(x)$ is the speed of the river current, g is the acceleration of gravity and h(x) is the depth of the water layer when undisturbed. Thus, $\rho F(x, y']$ is the energy density associated with the column of fluid whose horizontal dimensions are functions of x only, ρ = 1gmcm^3(app.). Thus, the functional $I(\Gamma)$ in equation (1.1) describes the entire mass transport between the two variable sides of the channel. This functional when evaluated along the extremal curve gives, the stationary value of the power associated with mass transport across the river. This power determines, to some extent, the intensity of the marine activities taken place across the channel.

In the fig. 1 of section 2, the mass flow prevails independent of the existing functional relationship among the parameters defining the geometry of the channel. It is, therefore, calculated that the mass transport velocity of 11.2msec^{-1} is sufficient to move a considerable load of materials across the river bed.

In the fig.2 of section 3, the situation is quite different; the constraining factor being the established inequality $R < c < 7R$. Here, R is the radius of the circular end arc and c is the focal distance of the hyperbolic end boundary from an assigned origin. Consequently, calculations strongly suggests that the data R = 5 km and c = 8 km are the least values sufficient for the initiation of the effective mass transport in such model channel.

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