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INVENTORY REPLENISHMENT POLICY WHEN DEMAND IS QUADRATIC: DIRECT AND INVERSE METHODS

F. E. U. OSAGIEDE AND S. E. OMOSIGHO
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BENIN,
BENIN CITY, NIGERIA

ABSTRACT

We present two new methods for solving the inventory replenishment problem when demand is given by $f(t) = bt^2$ over a finite time horizon H. No shortage is allowed in the system. The replenishment times t_i, i = 0, 1, 2,…, n - 1, where

$$H > t_{n-1} > t_{n-2} > \ldots > t_1 > t_o = 0,$$

are obtained and the order quantity at each replenishment time is derived. Example are given to illustrate these methods

Keyword: Inventory model, Replenishment times, Quadratic demand.

INTRODUCTION

This paper presents two new methods for solving the inventory replenishment problem when demand is $f(t) = bt^2$ and the time horizon H is finite. Further, no shortage is allowed. Since the publication of the work of Donaldson (1977), a number of papers have appeared in the open literature on inventory problems when demand is not constant. Most of these papers suggest the need to examine demand patterns different from constant demand. For example, Ritchie (1984) stated that "in practice many items held in inventory experience a period of rising demand". Hill (1995) further demonstrated the need to investigate inventory problems with rising demand. Hence in this paper, we examined the inventory model with quadratic demand pattern $f(t) = bt^2$. Two new methods are proposed for solving this problem. The first method is a direct analytic method and the second method is an inverse analytic method. In the first method, we need the number of replenishment in order to determine the replenishment points. In the inverse analytic method, we do not need to determine, n, the number of replenishment separately, it comes out naturally from the procedure.

THE PROBLEM AND NOTATION.

The inventory system examined in this paper is as follows. The demand for items is given by $f(t) = bt^2, b > 0, t \in [0, H], H < \infty$. The cost of replenishment per lot is c_1 and the holding cost per unit time is c_2. The time horizon, H is finite and no shortages are allowed. The inventory is zero at time 0 and at time H. The problem is to determine the optimal times t_i, $i = 0, 1, 2, \ldots, n-1$, at which to re-order so as to minimize the total relevant cost over [0,H]. We also wish to specify the re-order quantity at each re-order point. The total relevant cost, W, over [0,H] is given by

$$W = nc_1 + c_2 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (u - t_i) f(u)\, du \qquad (1)$$

For each fixed n, the necessary condition for optimal t_i,s, is given by

$$(t_i - t_{i-1}) f(t_i) - \int_{t_i}^{t_{i+1}} f(u)\, du = 0 \qquad (2a)$$

See Donaldson (1977)

If $f(t) = bt^2, b > 0$, *equation* (2a) *becomes*

$$\frac{1}{3} t^3_{i+1} = \frac{4}{3} t_i^3 - t_{i-1} t_i^2 \qquad (2b)$$

with $z_{i+1} = \frac{t_{i+1}}{t_i}$, *and* $z_i = t_i / t_{i-1}$, equation (2b) becomes

$$z_{i+1}^3 = 4 - \frac{3}{z_i} \qquad (3)$$

The problem is to determine t_1 and hence t_i, $i \geq 2$. First, we present a direct analytical method for solving this problem. The direct analytical methods is in line with the method proposed by Donaldson (1977) for the case of demand given by $f(t)$ = a + bt. In this method, we need n, the number of replenishments in order to determine the replenishment times.

THE DIRECT ANALYTICAL METHOD

Let $g_i = \frac{t_i}{t_1}$. It is obvious that $g_1 = 1$ and

$$g_i = \prod_{r=2}^{i} z_r, i = 2\ldots,n \qquad (4a)$$

Clearly $g_{i+1} = g_i z_{i+1}$, i = 1,2,3,…, n-1 (4b)

With t_n = H we have

$$t_i = \frac{g_i}{g_n} H \quad i = 2,3,\ldots,n \qquad (5)$$

Using $f(t) = bt^2$, equation (1) becomes

$$W = nc_1 + bc_2 \sum_{i=0}^{n-1} \left(\frac{t_i^4}{12} - \frac{t_i t_{i+1}^3}{3} + \frac{t_{i+1}^4}{4} \right. \qquad (6)$$

which reduces to

$$W = nc_1 + bH^4 c_2 g_n^{-4} \sum_{i=0}^{n-1} \left[\frac{1}{12} g_i^4 - \frac{1}{3} g_i g_{i+1}^3 + \frac{1}{4} g_{i+1}^4 \right] \tag{7}$$

using equation (5). Equation (6) and (7) are two equivalent expressions for the total relevant costs for the inventory problem. However, equation (6) depends on the replenishment points explicitly, whereas the evaluation of equation (7) does not need the replenishment points. Since the cost function is convex, we can evaluate equation (7) for various values of n and hence determine the optimum number of replenishments. Therefore, to proceed further, we evaluate equation (7) for various values of *n* and hence determine n which gives the minimum value of W. This value of *n* is optimal and for optimal n, it is easy to determine t_1 and hence t_i, $i \geq 2$ using equation (5). The order quantity at time t_i, $i = 0, 1, 2, \ldots n - 1$ is given by

$$Q_i = \int_{t_i}^{t_{i+1}} f(u)du = \frac{b}{3g_n^3}\left(g_{i+1}^3 - g_i^3\right) \tag{8}$$

The direct analytical method is summarized in the following algorithm:

ALGORITHM

Step 1: Determine $z_r, r \geq 2$ from $z_2 = \sqrt[3]{4}, z_{r+1} = \sqrt[3]{4 - \left(3/z_r\right)}$

Step 2: For each integer n=1,2...calculate $g_1, g_2 \ldots, g_n$ using $g_1 = 1$ and $g_{i+1} = g_i . z_{i+1}$ and then determine *W(n)*, where:

$$W(n) = nc_1 + bH^4 c_2 g_n^{-4} \sum_{i=0}^{n-1} \left[\frac{1}{12} g_i^4 - \frac{1}{3} g_i g_{i+1}^3 + \frac{1}{4} g_{i+1}^4 \right] \tag{9}$$

Step 3: Repeat step 2 until *W(n-1)* > *W (n)* and *W(n)* < W (n+1). The minimum cost is W(n) and *n* is optimal

Step 4: Determine $t_i, i = 1,2,\ldots, n-1$ using $t_i = g_i H/g_n$. For each t_i the order quantity is given be equation (8).

This method requires the evaluation of g_i starting from $z_2 = \sqrt[3]{4}$ and using the iterative scheme given in step 2. Further, numerous values of the cost function which are not needed will also be calculated. This is a major criticism of this method. In fact, the work of Donaldson (1977) has also been severely criticised on ground that it requires the use of a table. Hence, we propose the next method.

THE ANALYTIC INVERSE METHOD

Let $\{0, t_1, t_2, \ldots t_{n-1}\}$ be replenishment points and $t_n = H$. We assume that in the last interval $[t_{n-1}, t_n]$, the inventory holding cost is equal to the set - up cost i.e. $c_1 = c_2 I(t_{n-1})$ where

$$I(t_{n-1}) = c_2 \int_{t_{i-1}}^{t_i} (u - t_{i-1}) f(u) du \quad (10)$$

This gives

$$c_1 = bc_2 \left[\frac{t_{i-1}^4}{12} - \frac{t_i^3 t_{i-1}}{3} + \frac{t_i^4}{4} \right] \quad (11)$$

With $i = n$ and $t_n = H$, equation (11) becomes

$$\frac{t_{n-1}^4}{12} - \frac{H^3}{3} t_{n-1} + \left\{ \frac{H^4}{4} - \frac{c_1}{bc_2} \right\} = 0 \quad (12)$$

Using an iterative scheme such as Newton-Raphson method gives minimum real root of equation (12) for t_{n-1}. This gives the last replenishment point. The next replenishment point t_{n-2} is obtained as follows: we use $z_n = t_n / t_{n-1}$ to obtain z_n. From equation (3), we have

$$z_i = \frac{3}{4 - z_{i+1}^3} \quad (13)$$

If we set i = n-1 in equation (13), we obtain z_{n-1} and hence t_{n-2}. Arguing in this manner and making necessary changes we obtain t_{n-3}, t_{n-4},… until we have $t_{n-k+1} \geq 0$ and $t_{n-k} < 0$. The replenishment points are given by 0, t_{n-k+1}, t_{n-k+2} …, t_{n-1}.

COMPUTATIONAL RESULTS

In this section we present two examples to demonstrate the methods given above. Table 1 shows some values of z_n, g_n, and TW (n) where

$$TW(n+1) = \frac{1}{12} g_n^4 - \frac{1}{3} g_n g_{n+1}^3 + \frac{1}{4} g_{n+1}^4$$

n	z_n	g_n	TW(n+1)= $\frac{1}{12} g_n^4 - \frac{1}{3} g_n g_{n+1}^3 + \frac{1}{4} g_{n+1}^4$
1	-	1	0.3374011
2	1.5874011	1.5874011	0.3593063
3	1.2826326	2.0360524	0.3685394
4	1.1843003	2.4112975	0.3735401
5	1.1362208	2.7397663	0.3766552
6	1.1078415	3.0352268	0.3787755
7	1.0891584	3.3058428	0.3803085
8	1.0759461	3.5569087	0.3814666
9	1.0661175	3.7920826	0.3823728

Table 1: Values of z_n, g_n, and TW (n)

The values in Table 1 are used in examples 1 and 2.

EXAMPLE 1.

In this example, $f(t) = 100t^2, c_1 = 9; c_2 = 2/3, H = 1, b = 100.$

First, we use the analytic inverse method. Equation (12) yields the polynomial:

$$P(t_{n-1}) = 50t_{n-1}^4 - 200t_{n-1} + 69 = 0$$

which has the root 0.3486956363428. It is easy to verify that t_{n-2} is less than zero. Hence the replenishment points are 0 and 0.348695963. The Total Cost using equation (11) and these two replenishment points is W=27.25 units.

Next we use the direct method. Table 2 show the total relevant cost for n=1,2 and 3,

n	W(n)
1	25.667
2	24.167*
3	30.708

Table 2: Values of ***W(n)*** for different values of n.* indicates minimum cost.

From Table 2, it is evident that the optimum number of replenishment is 2. Using n=2, we have the replenishment points as 0 and 0.62996 and the associated total cost is 24.167 unit which is lower than the previous value given by the analytic inverse method.

EXAMPLE 2

Solve $f(t) = 100t^2, c_1 = 9, c_2 = 2/3, H = 3, b = 100.$

Using the analytic inverse method, we find that the required polynomial is

$$P(t_{n-1}) = 100t_{n-1}^4 - 10800t_{n-1} + 24138 = 0$$

with minimum real root t_{n-1}= 2.82334482398736. Continuing in the manner stated in the analytic inverse method above, i.e. repeated use of $z_n=t_n/t_{n-1}$ and starting from $t_n = 3$, $t_{n-1} = 2.82334482398736$; we have Table 3:

k	t_{n-k}	$z_{n-k+1}=t_n/t_{n-1}$
1	2.8233448	1.0625695
2	2.6354058	1.0713131
3	2.4337457	1.08286
4	2.2149146	1.0987989
5	1.9737512	1.1221853
6	1.701921	1.1597196
7	1.3843627	1.2203895
8	0.98839	1.4006239
9	0.41259662	2.3955358
10		-0.3077874

Table 3

From Table 3, we conclude that we have ten replenishment points. The replenishment points are as follows 0,0.4126, 0.9884, 1.3844, 1.7019, 1.9738, 2.2149, 2.4337, 2.6354, 2.8233. Further, the total cost using equation (11) is $W = 167.96$ units.

Using the Direct analytical method for the problem we obtain Table 4 which shows the associated total cost for various values of n. From Table 4, it is evident that n = 10 gives the minimum cost. Using $t_i = 3g_i / g_{10}$,we obtain the following replenishment points: 0, 0.7473814, 1.1863941, 1.5217078, 1.802159, 2.0476505, 2.2684722, 2.4707255, 2.6583675 and 2.8341321.

Thus both methods (Analytic inverse method and the Direct analytical method) gave ten replenishments with a total cost of 167.96 units and 164.6720 units respectively.

n	W(n)
1	1359.000
2	517.553
3	324.477
4	246.086
5	206.851
6	185.295
7	173.580
8	164.342
9	164.763
10	164.672*
11	166.323

Table 4: Values of the total relevant cost for various values of n
Minimum cost is obtained when n=10.

DISCUSSION AND CONCLUSION

The two methods are easy to apply but the direct analytic method requires the evaluation on numerous terms that may not be needed. This is a criticism of this method. A mixture of the two methods can also be implemented as follows First, the inverse method is used to obtain the number of replenishments, n. By evaluating the total relevant cost using equation (7) for $\boldsymbol{n-1}$, $\boldsymbol{n}$ and $n+1$ replenishments, it will be easy to determine if $\boldsymbol{n}$ is the optimum number of replenishments. The inventory replenishment problem when demand is given $f(t) = bt^k$ where $k \neq 1,2$ is currently been investigated.

REFERENCES

Donaldson W. A. (1977) Inventory Replenishment Policy for a linear trend in Demand-An Analytical Solution. Opl. Res. Q. Vol 28, pp 663-670.

Phelps R. I. (1980) Optimal Inventory rule for a linear trend in demand with a constant replenishment period J. Op! Res. Soc. Vol 31, pp 349-442.

Taha H. A. (1996) Operations Research-An Introduction.6th Edition. Prentice Hall Int. Inc London.

Richie E. (984) The E.O.Q. for linear increasing demand: A simple Optimal Solution. J. Opl. Res. Soc Vol 35,pp 949-952

Hill R. M. (1995) Inventory model for increasing demand followed by level demand. J Opl. Res. Soc Vol. 46,pp. 1250-1259