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PULSATING FLOW IN VISCOELASTIC TUBES

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ABSTRACT

The arteries and other blood vessels have been found to exhibit viscoelastic behaviour when blood flows through them. In this consideration we found the numerical solution of the velocity and pressure pulses in viscoelastic tubes. Increase in frictional term in viscoelastic tubes showed a considerable change in pressure and velocity pulses. The effects of tapering of the vessels were also investigated.

1. INTRODUCTION

At any location in the cardiovascular system, the motion of blood is driven by the local pressure gradient, which is in turn determined by the propagation of pressure pulse. Thus the fluid mechanical problems of the circulation can be divided into two parts: (1) an explanation of the pressure wave forms and (2) an analysis of the blood motion driven by the pressure gradient. However the geometric and physical properties of the blood vessel influence the pressure gradient and velocity wave forms as they develop at different points in the circulatory system (McDonald 1974). Careful study of these wave-form patterns may lead to a better understanding of the cardiovascular system especially under pathological condition in which changes in the mechanical properties of the vessel modify the pressure and velocity waveforms. The mechanical properties of a material depend not only on its composition but also on its structure and ultra structure (Bergel, 1972; Fung 1972). Biological tissues are not just elastic, the history of strain affects the stress. In particular, there is a considerable difference in stress response to loading and unloading. Experiments have shown that the stress-strain relationship of the carotid artery of a dog is not linear as expected of a Hookean material. It is not even a single curve (Fung 1978). This suggests an analog with the conventional theory of plasticity. However the tissue behaviour is more complex than that of an ideal plastic because:

1. the stress in both loading and unloading process are highly non-linear with respect to strain.
2. Pronounced stress relaxation and creep exist.

The behaviour of arteries and other blood vessels depends on the dynamic nature of the stress imposed on them with the resulting response being termed *viscoelastic*. Purely elastic materials strain instantaneously when stressed with the strain dependent only upon the magnitude of the applied stress. Energy is stored and recovered without loss. In contrast, viscoelastic materials have a retarded response to a stress situation and a consequent loss of energy to the surrounding tissue as heat. Part of the deformation is associated with the internal material viscosity and is dependent upon the rate at which the stress is applied. Thus for a viscoelastic materials, the strain depends upon both the magnitude of the stress and the rate at which it is applied. In the case of blood vessels, viscoelastic behaviour is well shown. After a sudden stretch, tension rises and decays towards some final value, stress relaxation (Stacy, 1957, Skalak, Ozkaya and Skalak 1989). Following a sudden change in tension, a continuous deformation (creep) is also observed. We consider the arteries as viscoelastic tubes.

2. MATHEMATICAL FORMULATION

In this work, we adopt a quasi-linear one dimensional model for an incompressible fluid in a distensible tube, based on the assumptions that (i) the wave length is long compared with the tube diameter and (ii) that the tube is constrained from longitudinal motion, and (iii) the wall of the material is assumed to be viscoelastic. Under these conditions, the governing equations are the equations of the continuity of mass and the conservation of momentum as used by many authors (Anliker et al, 1971, Smith and Hoogstraten, 1981, Oghre and Akinrelere, 1998)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}-f=0 \qquad (2.1)$$

$$\frac{\partial A}{\partial t}+\frac{\partial}{\partial x}(Au)+\Psi=0 \qquad (2.2)$$

Where A is the Area of the tube, u the average longitudinal velocity along the axial coordinate x. p the pressure, ρ the fluid density and f the frictional term. Ψ is the outflow function along the tube and t is the time coordinate.

If we ignore the outflow function Ψ and the frictional term f equations 1 and 2 are similar to the one dimensional hyperbolic equations of gas dynamics. Hence shock may develop in the solutions (Sachedev, et al, 1997).

The function Ψ and f are chosen to be known functions of A, u, P, x, and t so we have two differential equations in three unknowns: A, u, P. Therefore a third equation to complement them is needed. This is the state equation. For a purely elastic model, the state equation simply relates A and P which holds under static condition. However we are considering viscoelastic tube, the equation of state is the equation relating the pressure to the behaviour of the wall of the tube. According to Bergel (1961a,b), in his studies of the static and dynamic elastic properties of the arterial wall, viscous substance can be formally represented in the same way as purely elastic substances if the elastic modulus, the proportionality constant between stress and strain is considered to be a complex quantity. For a viscous material, this implies that a phase difference exists between the stress and the strain i.e the pressure and the radius variations are not in phase so the elastic and viscous moduli are respectively

$$E_{r} = E_{c}\cos\phi = E_{dyn} \qquad (2.3)$$

$$E = E_{c}\sin\phi = \eta w \qquad (2.4)$$

E_c is the dynamic incremental modulus, ϕ is the phase angle between pressure and radius variation and ω is the angular frequency. At zero frequency i.e. for a purely elastic material,

$$E_{dyn} = E_c, E_i = 0 \quad (2.5)$$

So $E_r = E_c$ is called the static modulus of elasticity and $\phi = 0$.

Bergel's experiment (1961b) on excise vessel segments uniformly shows pressure variation leading area variation for all frequencies. Also in vivo measurements in major arteries indicate generally the pressure leading the area. Therefore to formulate the state equation, we first assume that the solutions of the equations of motion and mass are periodic in time. The conversion of periodic wave forms such as pressure and flow pulsation to numerical form is done by Fourier or harmonic analysis (Womersley 1955). Thus we will model the viscoelastic behaviour of the arteries through Fourier decomposition of pressure wave form and phase correction to each harmonic in order to give the area of the tube. The state equation describes the response of the wall to the internal pressure. It relates the lumen area to the instantaneous pressure and time rate of change of pressure. For a purely elastic wall the area is a function of only the pressure. For most normal physiological pressure pulses, the static pressure-area is linear (Bergel 1961a) of the form

$$A = A_m\left(1 + \frac{P - P_0}{\rho c_0^2}\right) \quad (2.6)$$

A_m is the area, c_0 is the pulse wave speed corresponding to the mean proximal pressure P_0. We shall use this equation as a basis for the development of a viscous state equation. The viscous component of the blood vessel behaviour is represented by phase and angle shifts. Based on Bergel's (1961a,b) data, Saito & Werff (1975) using linear regression line gave

$$\phi_{lag} = \phi_0 + kf \quad (2.7)$$

ϕ_{lag} is the phase angle by which radius changes lag the pressure changes. ϕ_0 is the zero frequency intercept and f is the pulse frequency k is a constant. When physiological waveform are Fourier analysed, all frequencies are integral multiple of the fundamental frequency based on the heart rate. Thus for the nth harmonic, phase lag is:

$$\phi_{lag,n} = \phi_0 + \frac{nkw}{2\pi} \quad f = \frac{nw}{2\pi} \quad (2.8a)$$

$$\phi_0 + Knw, \quad K = \frac{k}{2\pi} \quad (2.8b)$$

Since area is directly related to radius, this equation also determines the phase angle by which the area variation lags behind the pressure variation.

To develop the viscoelastic state equation we consider the pressure curve that is a periodic function and is decomposed into its Fourier harmonic as

$$P = P_n + \sum_0^{\infty} P_n \sin(nwt + \phi_n) \quad (2.9)$$

P_n is the mean pressure, P_n is the magnitude of the nth harmonic and ϕ_n is the phase shift of the nth harmonic. The nth harmonic. The nth harmonic area is related to the same harmonic of pressure through the phase lag given by $\phi_{lag,n}$. By analogy with the static pressure area relation, the amplitude of the area harmonics are given by $A_m P_n/\rho c^2_0$. With this consideration the area presssure relation for a viscoelastic material becomes:

$$A = A_m\left[1 + \frac{P_m - P_n}{\rho c_0^2} + \sum_0^\infty \frac{P_n}{\rho c_0^2}\sin(nwt + \gamma_n)\right] \qquad (2.10)$$

where γ_n is the phase angle of the nth harmonic of area and is given by

$$\gamma_n = \phi_n - \phi_{lag,n}$$
$$= \phi_n - \phi_0 - Knw \qquad (2.11)$$

We see that the term Knw simply shifts the area curve uniformly in time by an amount K but the ϕ_0 term shifts each harmonic differently in time resulting in a change of shape of the area waveform.

Equation 2.10 is our viscoelastic state equation and we solve this together with equations 2.1 and 2.2. We now specify the geometry and behaviour of the arteries:

(i) **Tapering**
The cross sectional area of the individual arteries decreases as one proceeds away from the heart. This axial taper of lumen area is postulated to be a decaying exponential function of x and we have

$$A_m(x) = A_0 \exp(-\beta x/R_0) \qquad (21.12)$$

A_0 and R_0 are mean proximal area a and radius and β is the taper factor.
(ii) **Arterial denstensibility** of the vessel is specified implicitly in the model by the pulse wave speed. For a purely elastic thin wall vessel containing an inviscid fluid with no wave reflection, this relationship is given by the familiar Moens-Korteweg equation

$$c_0^2 = \frac{Eh}{2\rho R} \qquad (2.13)$$

(where E is the circumferential Young modulus, R the internal radius, h the wall thickness and ρ the wall density). Since h/R is constant, the pulse wave speed is proportional to the square root of the elastic modulus. This equation can be corrected for the fihite wall thickness by applying Bergel's formula (1961b) with a Poisson's ratio σ of 0.5. Thus:

$$c_0^2 = \frac{Eh}{2\rho R(1-\sigma^{-3})} = \frac{2}{3}\frac{Eh}{\rho R} \qquad (2.14)$$

However it is not possible to explicitly determine the instantaneous wave speed from equation 2.11, so we use the static area-pressure relation (2.6). This is justified by the fact that there should be no measurable influence of wall viscosity on the pulse wave speed in the physiological range (Hardung 1963. Maxwell and Anliker 1968, Remington, 1967). Thus the instantaneous wave speed is given as:

$$c = c_0\sqrt{A/A_m} \qquad (2.17)$$

which gives wave speed above the mean wave speed c_0 for area and pressure above the mean as physically observed by Anliker et al (1971).
Thus we have

$$\frac{\partial A}{\partial t}=\frac{\partial A}{\partial P}\frac{\partial P}{\partial t}=\frac{A_m}{\rho c_0^2}\frac{\partial P}{\partial t}$$

$$\frac{\partial A}{\partial x}=\frac{A_m}{\rho c_0^2}\frac{\partial P}{\partial x}+\left(1+\frac{P-P_0}{\rho c_0^2}\right)\frac{\partial A_m(x)}{\partial x}$$

$$=\frac{A_m}{\rho c_0^2}-\frac{\beta A}{R_0} \qquad (2.18)$$

Equation 2.1 (the continuity equation) becomes,

$$\frac{\partial P}{\partial t}+\frac{u\partial P}{\partial x}+\rho c^2\left(\frac{\partial u}{\partial x}-\frac{\beta u}{R_0}+\frac{\Psi}{A}\right)=0 \qquad (2.19)$$

Equation of motion is

$$\frac{\partial u}{\partial t}+\frac{u\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}-f=0 \qquad (2.20)$$

subject to the boundary condition at the proximal site giving as,

$$u(0,t)=u_0+\sum_{n=1}^{10}U_n\sin(nwt+\phi_n') \qquad (2.21)$$

$$P(0,t)=P_0+\sum_{n=1}^{10}U_n\sin(nwt+\phi_n) \qquad (2.22)$$

The slope of the characteristic directions according to equations (2.19 and 2.20) are given as:

$$\frac{dx}{dt}=u+c \qquad (2.23)$$
$$\frac{dx}{dt}=u-c$$

The relationship between u and P along these two directions are:

$$du+\frac{1}{\rho c}dP=\left(f+\frac{c\beta u}{R_0}-\frac{c\Psi}{A}\right)dt \qquad (2.24)$$

$$du-\frac{1}{\rho c}dP=\left(f-\frac{c\beta u}{R_0}+\frac{c\Psi}{A}\right)dt$$

3. NUMERICAL CALCULATIONS

First order finite difference approximation (where subscript a means evaluated at a) gives

$$x_p - x_r = (u+c)_c(t_p - t_r) = (u+c)_c(t_c - t_r) \quad \frac{\Delta x}{(u+c)_c} = t_c - t_r$$

$$t_r = t_r - \frac{\Delta x}{(u+c)_c}$$

$$t_s = t_c - \frac{\Delta x}{(u-c)_c} \tag{3.1}$$

$$\frac{\Delta x}{\Delta t} = \frac{pc}{bc}$$

$$(u+c)_c = \frac{pc}{rc}$$

$$\frac{(u+c)_c}{\Delta x/\Delta t} = \frac{pc}{rc} \div \frac{pc}{bc} = \frac{bc}{rc}$$

By interpolation

$$\frac{u_b - u_c}{u_r - u_c} = \frac{(u+c)}{\Delta x/\Delta t} = \frac{\Delta t}{t_c - t_r}$$

$$u_r = u_c + (u_b - u_c)\frac{(t_c - t_r)}{\Delta t}u_b$$

Similarly

$$= u_c\left[1 - \frac{t_c - t_r}{\Delta t}\right] + \left(\frac{t_c - t_r}{\Delta t}\right)u_b \tag{3.2}$$

$$P_r = P_c\left[1 - \frac{t_c - t_r}{\Delta t}\right] + \left(\frac{t_c - t_r}{\Delta t}\right)P_b. \tag{3.3}$$

Also for the negative slope

$$\frac{\Delta x}{\Delta t} = \frac{pc}{ca}$$

$$(u-c) = \frac{pc}{sc}$$

$$\frac{u-c}{\Delta x/\Delta t} = \frac{ca}{sc} = \frac{u_c - u_a}{u_s - u_c}$$

$$\frac{u_c - u_a}{u_s - u_c} = \frac{\Delta t}{t_c - t_s}$$

$$u_s = u_c + (u_c - u_a)\left(\frac{t_c - t_s}{\Delta t}\right)u_a$$

$$u_s = u_c\left[1+\frac{t_c - t_s}{\Delta t}\right] - \left(\frac{t_c - t_s}{\Delta t}\right)u_a \qquad (3.4)$$

Similarly
$$P_s = P_c\left[1+\frac{t_c - t_s}{\Delta t}\right] - \frac{t_c - t_s}{\Delta t}P_a \qquad (3.5)$$

Integrating equations (3.2-3.5) by finite difference approximation along the two characteristic directions we have

$$\begin{aligned} u_p - u_r + \frac{1}{(\rho c)_c}\left(P_p - P_r\right) &= \left[f + \frac{c\beta\Psi}{R_0} - \frac{c\Psi}{A}\right]_c \left(t_p - t_r\right) \\ u_p - u_s - \frac{1}{(\rho c)_c}\left(P_p - P_s\right) &= \left[f - \frac{c\beta\Psi}{R_0} + \frac{c\Psi}{A}\right]_c \left(t_p - t_s\right) \end{aligned} \qquad (3.6)$$

Adding the two equations we have

$$2u_p = u_r + u_s + \frac{1}{(\rho c)_c}\left(P_s = P_p\right) + f_p\left(2t_c - t_r - t_a\right) + \left(\frac{c\beta\Psi}{R_0} - \frac{c\Psi}{A}\right)\left(t_s - t_s\right) \quad (3.7)$$

Subtracting the first of equation 3.6 from second equation we have:

$$\frac{2}{\rho c}P_p = \frac{1}{\rho c}\left(P_r + P_s\right) + \left(u_r - u_r\right) + \left(\frac{c\beta\Psi}{R_0} - \frac{c\Psi}{A}\right)\left(2t_p - t_r - t_c\right) + f\left(t_s - t_r\right) \quad (3.8)$$

The computational method adopted here is the method of specified time interval as outlined by Lister (1960). We non-dimensionalized the time interval by dividing by the period of one cycle which is 0.38745s. Then the grid runs from 0 to 1 in the t direction and from 0 to some terminal value L in the x direction. The time interval T = 1/N where N is the number of segments between 0 and 1. The distance interval Δx = 1/N where L is the length of the aortic interval and N is the number of segment between 0 and L. The constraint on the choice of these interval is that the characteristics must fall on the line segment and we assume this condition has been met in our computation.
Equations 3.1 and 3.9 can now be written in the computational forms as:

$$t_{RN} = NT - \frac{\Delta x}{u_N + c_N} \qquad (3.9)$$

$$t_{SN} = NT - \frac{\Delta x}{u_N - c_N} \qquad (3.10)$$

$$u_{RN} = u_N\left[1 - \left(\frac{NT - t_{RN}}{T}\right)\right] + \left(\frac{NT - t_{RN}}{T}\right)u_{N+1} \qquad (3.11)$$

$$u_{SN} = u_N\left[1+\left(\frac{NT-t_{SN}}{T}\right)\right]-\left(\frac{NT-t_{SN}}{T}\right)u_{N-1} \quad (3.12)$$

$$P_{RN} = P_N\left[1-\left(\frac{NT-t_{RN}}{T}\right)\right]+\left(\frac{NT-t_{RN}}{T}\right)P_{N+1} \quad (3.13)$$

$$P_{SN} = P_N\left[1-\left(\frac{NT-t_{SN}}{T}\right)\right]-\left(\frac{NT-t_{SN}}{T}\right)P_{N-1} \quad (3.14)$$

$$u_{PN} = 0.5\left(u_{RN}+u_{SN}-\frac{1}{c_0\rho}\right)\left(P_{SN}-P_{RN}\right)+$$

$$\left(f_N-\frac{u_{N-1}}{A_0}\right)\left(2t_N-t_{RN}-t_{SN}\right)+\left(\frac{c_N\beta u_N}{R_0}-\frac{c_{N-1}}{A_N}\right)\left(t_{SN}-t_{RN}\right) \quad (3.15)$$

$$P_{PN} = 0.5c_N\left(\left(u_{RN}-u_{SN}\right)+\frac{1}{\rho c_N}\left(R_{RN}+P_{SN}\right)+\right.$$

$$\left(F_N-\frac{c_N u_{N-1}}{A_N}\right)\left(t_{SN}-t_{RN}\right)+\left(\frac{c_N\beta u_N}{R_0}-\frac{c_{N-1}}{A_N}\right)\left(2t_N-t_{RN}-t_{SN}\right)$$

$N = 1 : 100$ (3.16)

The initial condition on velocity and pressure can now be put in the form

$$u_N(0,t) = U_0+\sum_{n=1}^{10}U_n\sin\left(nwNT+\phi_n\right) \quad (3.17)$$

$$P_N(0,t) = P_0+\sum_{n=1}^{10}P_n\sin\left(nwNT+\phi_n\right) \quad (3.18)$$

The equations for the frictional term, the arterial area, the wave speed are now respectively:

$$F_N = \frac{8vu_N}{R_0^2} \quad (3.19)$$

$$A_N = A_0\left(1+\frac{1}{\rho c_0}\sum_{n=1}^{10}P_n\sin\left(nwNT+\phi_n-\phi_0-nkw\right)\right) \quad (3.20)$$

$$c_N = c_0\sqrt{|A_N|A_0} \quad (3.21)$$

Three old grids are needed to calculate the value of u P at each new grid point. However at the boundary points of each line $x = x_0 + \Delta x$ only two grid points are available and the calculation will break down. This is when N = 1 and N = 100. We overcome this by

invoking the periodicity of solution which is the essential advantage of the periodic method of characteristics. Thus at the beginning of each line segment calculation, this is stated. The calculation mode consists of NT + 2 modes in the + direction with (N-1)T=0 corresponds to N=1 at t=0 and 100T = 1 corresponds to t = 1. The periodicity condition says that the values at t = 0 and t = 1 or (N = 1)T and NT must be same. Also the values at IT and 101T must also be the same. With these we have 102 modes which allows us to proceed in our calculation matching forward in space. Thus at each calculation segment the two boundary conditions must first be specified before proceeding on the calculation. Here lies ht advantage of the periodic method of characteristics over the ordinary method of characteristics. The two specify that two boundary conditions must be given one at the proximal end, the other at the distant end. However, the proximal boundary condition in the periodic method is not predetermined but is deduced from that of the proximal conditions.

The other problem was the specification of the initial values of the pressure and velocity. We thus discretised Vander Werff (1974) experimental wave forms to obtain values of the velocity and pressure. The values for 20 readings for both velocity and pressure are in Tables II and III.

PROPERTIES	NOTATION	VALUE
Proximal aortic radius	R_0	0.945
Length of tube	L	14.2
Period for one cycle	T	.3874
Mean proximal pressure	P_0	119.1
Mean proximal velocity	U_0	16.5
Proximal aortic area	A_0	
Taper factor	β	0.0367
Density of blood	ρ	1.055
Angular frequency	ω	16.23
Viscoelastic component	ϕ_0	0.0753
Wave speed	c_0	500

TABLE I: INPUT DATA

U	5	6	12	50	80	88	60	38	0
T	195	220	260	265	270	315	345	340	360
U	0	6	3	0	1	4	4	5	0

TABLE II: INPUT DATA OF VELOCITY AND TIME

T	0	10	18	35	48	70	90	140	145	150
P	103	102	103	135	140	135	130	120	119	120
T	165	180	195	225	235	26	288	296	330	360
P	122	118	120	123	116	115	112	109	110	103

TABLE III: INPUT DATA OF PRESSURE AND TIME

We have that if

$$u(0,t) = U_0 + \sum_{n=1}^{10} U_n \sin(nwt + \phi_n)$$

$$\equiv U_0 + \sum_{n=1}^{10} A_n \sin(nwt) + \sum_{n=1}^{10} B_n \cos(nwt) \qquad (3.22)$$

$$P(0,t) = P_0 + \sum_{n=1}^{10} P_n \sin(nwt + \phi_n)$$

$$\equiv P_0 + \sum_{n=1}^{10} A_n \sin(nwt) + \sum_{n=1}^{10} B_n \cos(nwt) \qquad (3.23)$$

We need to determine the values of U_n's P_n's, ϕ_n's and ϕ_n's
From equation 3.23 we have that

$$U_n = \left(A_n^2 + B_n^2\right) \qquad (3.24)$$

$$\phi_n = \tan^{-1}\left(B_n / A_n\right) \qquad (3.25)$$

Using our discretised data, we solve a 20 x 20 matrix using Matlab 4.0 Simulink and equations 3.34 and 3.25 to determine the values of u_n's P_n's, ϕ_n's and ϕ_n's.

The viscoelastic property is determined from the value of K, and ϕ_0 in equation 2.11. To measure this effect, we increase the values of K and ϕ_0 simultaneously in steps of 2K, 5K, 10K, 20K corresponding with $2\phi_0$, $5\phi_0$, $10\phi_0$, $20\phi_0$ to find the effect on the values of u and P. We also decrease K in the steps of 0.1K, 0.01K 0.001K and 0.0001K correspondingly with $0.1\phi_0$, $0.015\phi_0$, $0.001\phi_0$, $0.0001\phi_0$.

4. RESULT ANDCONCLUSION

To measure the effect of friction and tapering, for each value of K and ϕ_0 along with line segment, we compute u and P when the frictional term and the tapering coefficient are varied and compared the result with our input data.

An increase in the viscoelastic coefficients does not produce any significant change in the pressure and velocity wave form. However when the frictional term was increased in a viscoelastic tube there was a considerable change in the pressure and flow pulses especially towards the distal end of the tube. The increase in friction causes a rapid decay of pressure and velocity wave form. We also found out that an increase in the cross sectional areas causes lower flow velocity and pulse pressure. This must be due to the fact that the same amount of liquid flows through the larger arteries and the pulse pressure is decreased as a result of increasing the distensibility of the artery.

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