Iig. Ass. Math. Phys.
l. 4 (2000)

ON THE EXISTENCE AND UNIQUENESS OF SELF SIMILAR DIFFUSION EQUATIONS

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ABSTRACT

We consider the diffusion equation

$$u_t = \nabla \cdot (u^n \nabla u) + \delta |x|^{-\beta} u^m \quad \text{in } R^N \; X \; (0, \infty)$$

we determine the criteria for self-similar solutions of the form

$$u(x,t) = (T-t)^p \;\; f\,(|x|\,(T-t)^q)$$

1. INTRODUCTION

The equation $u_t = \nabla \cdot (u^n \nabla u) + \delta |x|^{-\beta} u^m$ (1)

arises in many areas of application. The case n = δ = 0 corresponds to the classical heat equation. When n > 0 (n< 0) and δ = 0, the equation model shows (fast) flow of a gas in a porous medium. When δ ≠ 0 equation (1) models microwave heating of various materials see [1] and [2].

Peletier and Zhang [2] considered equation (1) when

$$\delta = 0 \text{ and } u(x,0) = u_0(x) \text{ for } x \, \varepsilon R^N \tag{2}$$

Where the initial distribution u_o is non-negative and $u_o \, \varepsilon L^1(R^N)$. they investigated self-similar solution of the form

$$u(x,t) = (T-t)^p \quad f(|x| \; (T-t)^q) \tag{3}$$

where p > 0 and q ε R are constants. By symmetry

$$f' = 0 \text{ at } \eta = 0. \tag{4}$$

Peletier and Zhang [2] require f to represent a solution of equation (1) of finite mass and this implies

$$\int_0^\infty \eta^{N-1} f(\eta) d\eta < \infty \tag{5}$$

that is

$$f(\eta) \approx \eta^{-\frac{N-2}{1-n}} \text{ as } \eta \to \infty \tag{6}$$

where N > 2 and $\frac{2}{N}$ <n < 1. For δ = 0.

The above conditions lead to the system of equations

$$(p)\begin{cases} \eta^{1-N}(\eta^{N-1} f^n f')' - q\eta f' + pf = 0, \; f > 0 \text{ for } \eta > 0 & (7) \\ f'(0) = 0 \text{ and } f(0) = 1 & (8) \\ f(\eta) \cong \eta^{-\frac{N-2}{1-n}} \quad \text{as } \eta \to \infty & (9) \\ pn + 2q = -1 & (10) \end{cases}$$

Peletier abd Zhang [2] showed that problem p has exactly one solution (f, ρ, q) for every

$$N>2 \text{ and } \frac{2}{N} < n < 1 \tag{11}$$

Ajadi [1] considered equation (1) when $\delta \neq 0$, he showed that there exists an analytical solution

$$f = \frac{(-n\eta^2)}{2[4 + 2nN + n^2\delta]} \tag{12}$$

when

$$\beta = 2, m = n + 1, q = 0, p = -1 \tag{13}$$

Due to non-zero flux at infinity Ajadi shows that u blows up or quenches in a finite time depending on whether $n > 0$ or $n < 0$.

2. SELF-SIMILAR SOLUTIONS

We put (3) into (1) to obtain

$$ff'' + = \frac{N-1}{\eta} ff' + \frac{1}{n}(f')^2 - qnf' - pf + n\delta\eta - \beta f\frac{m}{n} - \frac{1}{n} + 1 \tag{14}$$

where

$$p = \frac{(2-\beta)}{\beta - \frac{2m}{n} + . \frac{2}{n}} \tag{15}$$

$$q = \frac{\frac{m}{n} - \frac{1}{n} - 1}{\beta - \frac{2m}{n} + \frac{2}{n}} \tag{16}$$

thus

$$p + 2q + 1 = 0 \tag{17}$$

In this paper and as in [2] we require finite mass unlike [1] but assume $\delta \neq$ as in [1] ($\delta = 0$ in [2]).

We therefore consider the problem

$$
(p)\begin{cases}
ff'' + \frac{N-1}{\eta}ff' + \frac{1}{n}(f')^2 - qnf' - pf + n\delta\eta^{-\beta} f \frac{m}{n} - \frac{1}{n} + 1 \\
\text{for } \eta > 0 & (18) \\
f'(1) = 0 \text{ and } f(1) = 1 & (19) \\
f(\eta) = \eta^{-\frac{N-2}{1-n}} \quad \text{as } \eta \to \infty & (20) \\
p + 2q = -1 & (21)
\end{cases}
$$

THEOREM: For every

(i) $N > 2$ and $\frac{2}{N} < n < 1$ and

(ii) $\frac{m}{n} - \frac{1}{n} \geq 1$, $\beta \geq -1$ and $q = 0$

problem P has exactly one solution (f,p,q).

PROOF: Let $x_1 = \eta$, $x_2 = f$ and $x_3 = f'$

Then we obtain

$$
\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ -\frac{(N-1)}{n}x_3 - \frac{1}{n}\frac{x_3^2}{x_2} - \frac{n\delta x_2}{x_1\beta}\frac{m}{n} - \frac{1}{n} - 1 \end{pmatrix} \tag{22}
$$

$$
\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{23}
$$

clearly,

$$
x_1 \varepsilon [1, \infty], x_2 \varepsilon (x_1^{-\frac{N-2}{1-n}}, 1)
$$

$$
x_3 \varepsilon [-\frac{N-2}{1-n} x_1^{-\frac{N-2}{1-n}-1}, 0] \tag{24}
$$

Hence

$\frac{\delta f_i}{\delta x_j}$, $i, j = 1,2,3$ is bounded .

The theorem follows.

REFERENCES

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