J. Nig. Ass. Math. Phys.
Vol. 4 (2000)

TOEPLITZ – CIRCULANT PRECONDITIONER FOR SOLUTIONS OF LARGE SYSTEMS OF LINEAR EQUATIONS.

FELIX OKORO
DEPARTMENT OF COMPUTER SCIENCE AND STATISTICS,
AMBROSE ALLI UNIVERSITY,
EKPOMA, NIGERIA.

ABSTRACT: This paper considers the solution of discretized 3-dimensional Poisson equation by constructing a circulant preconditioner. The method of solution of Toeplitz matrices is applied with the Conjugate Gradient Method to obtain the solution. By exploiting the Toeplitz structure of the discretized system, a fast convergence method is attained.

INTRODUCTION:

Large symmetric positive definite systems of linear equations of the form A*x = b are often generated in different areas of science and engineering. Such as finite element models of partial differential equations, signal processing, deconvolution, etc. Several numerical analysis techniques have been proposed to its solution. [2]
Recently, Strang [8] and Olkin [6] independently proposed using the Preconditioned Conjugate Gradient (PCG) method to solve symmetric positive definite Toeplitz Systems. Consequently, the design of effective Toeplitz preconditioners have received much attention. Similarly, the preconditioned Conjugate Gradient Method is an active area of research. One of the reasons for this is that the Conjugate Gradient Method (CGM) has been shown to be an effective iteration method. [5] , [3]. While the rate of convergence of the CGM is quite rapid, however, when the size of the matrix becomes large, the computing time for that number of iterations is usually prohibitive. The rate of convergence is known to be dependent on the distribution of the eigenvalue of the matrix. Let k_{min} and k_{max} be the smallest and largest eigenvalue of the matrix A, an estimate of the rate of convergence was given in [1] as

$$\left\| x_k - x^* \right\| \le 2\sqrt{k}\,\gamma^k \left\| x_0 - x^* \right\|_2$$

Where x^* is the exact solution of the system
$\gamma = (k-1)/(k+1)$ and k = (cond A) is the condition number of A in the L_2 norm. It is noted that $\gamma = 0$, when $k = 1$ and $\gamma \to 1$ as $k \to \infty$. Hence, the larger k, the closer γ will be to 1 and the slower will be the rate of convergence. This is the situation in many applications. Consequently, it is important to modify the equation and this leads to the idea of increasing the rate of convergence by preconditioning A.

2. THE MODEL PROBLEM

Mathematical Formulation

Our model problem is the 3-dimensional Poisson equation of the form

$$\nabla(K\nabla U) = F; \qquad (x, y, z) \in \Omega \qquad 2.1$$

Where k and F are given functions of the three spatial variables.
From (2.1), we have

$$\nabla K \cdot \nabla U + K\nabla^2 U = F \qquad 2.2$$

Assuming k is a constant, then (2.2) becomes

$$\nabla^2 U = \overline{F} ; \quad \overline{F} = F / K \qquad 2.3$$

If $\overline{F} \equiv 0$, we have the Laplace equation .Let u (x,y,z) represent the equilibrium temperature distribution in a 3 – dimensional heat – conducting medium Ω defined on a cube 0 < x,y,z < 1. To obtain a system of finite difference equation for (2.2), [4] we approximate the derivatives by the central difference scheme to get the following:

$$[\frac{\partial k}{\partial x} + \frac{\partial k}{\partial y} + \frac{\partial k}{\partial z}] \cdot \{\frac{U_{i+1,j,K} - U_{i-1,j,K}}{2h}$$

$$+ \frac{U_{i,j+1,K} - U_{i,j-1,K}}{2h} + \frac{U_{i,j,K+1} - U_{i,j,K-1}}{2h}\}$$

$$+k(x,y,z) \ \{\frac{u_{i-1,j,k} - 2u_{i,j,k} + u_{i+1,j,k}}{h^2}$$

$$\frac{u_{i,j-1,k} - 2u_{i,j,k} + u_{i,j+1,k}}{h^2} + \frac{u_{i,j,k-1} - 2u_{i,j,k} + u_{i,j,k+1}}{h^2}\}$$

$$= F(x,y,z) \qquad (2.4)$$

Where we have assumed equal grid size h on all dimension.
If $k(x,y,z) \equiv xyz$ and

$$F(x,y,z) \equiv \left(Sin\frac{\pi}{2}x + Cos\frac{\pi}{2}y + Z \right), \text{ then (2.4) becomes}$$

$$2(xyz) . \{U_{i-1,j,k} + U_{i+1,j,k} + U_{i,j-1,k} + U_{i,j+1,k} + U_{i,j,k-1} + U_{i,j,k+1} - 6U_{i,j,k}\}$$
$$+ h(yz + xz\,xy) \ \{U_{i+1,j,k} - U_{i-1,j,k} + U_{i,j+1,k} - U_{i,j-1,k} + U_{i,j,k+1} - U_{i,j,k-1}\} \qquad 2.5$$
$$= -2h^2 \left(Sin\frac{\pi}{2}x + Cos\frac{\pi}{2}y + z \right).$$

Defining approximations u_{ijk} to the exact solutions u(x,y,z) at the N^3 interior grid points, (2.5) gives

$$
\begin{aligned}
2(ijk)\ \{&-U_{i-1,j,k} - U_{i+1,j-1,k} - U_{i,j+1,k} - U_{i,j,k-1} - U_{i,j,k+1} + 6U_{i,j,k}\} \\
&+ h(jk+ik+ij)\ \{-U_{i+1,j,k} - U_{i-1,j,k} - U_{i,j+1,k} \\
&- U_{i,j-1,k} - U_{i,j,k+1} - U_{i,j,k-1} \qquad \text{2.6a} \\
&= -2h^2\left(Sin\frac{\pi}{2}x + Cos\frac{\pi}{2}y + z\right) \\
&i,j,k = 1, 2, \ldots\ldots\ldots\ldots\ldots, N
\end{aligned}
$$

If we assume the Dirichlet boundary conditions;

$$
\begin{aligned}
U(0,y,z) &= U(x,0,z) = U(x,y,0) = 0 \\
U(1,y,z) &= U(x,1,z) = U(x,y,1) = 0, \qquad \text{2.6b}
\end{aligned}
$$

equation (2.6) is therefore a system of N^3 equations in the N^3 unknown u_{ijk}, $i, j, k = 1, 2, \ldots\ldots, N$ corresponding to the interior grid points, $h = 1/N$.

Analysis

Equation (2.6) is written in matrix – vector form. For this purpose, the interior grid points are numbered in a form called the natural or row – wise ordering. Corresponding to this ordering of the grid points, we order the unknown u_{ijk} into the vector:

$$(U_{111}, U_{211} .., U_{N11};\ U_{121}, .U_{221}, .., U_{N21};\ U_{NN1}, U_{NN2}, U_{NNN}) \quad 2.7$$

In this wise, writing the system of equation (2.6) in block form, For N = 3, we have the coefficient matrix become:

$$
A_{nn} = \begin{bmatrix}
B & I & & I & & & & & \\
I & B & I & & I & & & & \\
& I & B & & & I & & & \\
I & & & B & I & & I & & \\
& I & & I & B & I & & I & \\
& & I & & I & B & & & I \\
& & & I & & & I & B & I \\
& & & & & & I & & I\ B
\end{bmatrix} \quad \text{2.8}
$$

Here, the blocks B are square matrices of order n and I denote the n x n identity matrix. Equation (2.6) becomes the N^3 x N^3 coefficient matrix in block tridiagonal matrix (2.8).

3. CONSTRUCTION OF CIRCULANT PRECONDITIONER

We state the following definitions that are needed in the sequel

Definition 3.1

Toeplitz matrices:

A matrices A = [aij] $\in M_{n+1}$ of the form

$$T=\begin{bmatrix} t_0 & t_1 & t_2 & - & - & - & t_n \\ t_{-1} & t_0 & t_1 & - & - & - & t_{n-1} \\ t_{-2} & t_{-1} & t_0 & - & - & - & - \\ & & & & & & \\ t_{-n} & t_{-n+1} & t_{-n+2} & - & - & - & t_0 \end{bmatrix}$$

is called a Toeplitz matrix. The general term Tij=Tj-i for some given sequence t-n,t-n+1,. . ., $t_{-1},t_0,t_1,t_2,$ . . . $t_{n-1},t_n \in C$. The entries of T are constant down the diagonals parallel to the matrix diagonal [7]

Definition 3.2

Circulant Matrices

A matrix A$\in M_n$ of the form

$$C=\begin{bmatrix} c_1 & c_2 & - & - & - & c_n \\ c_n & c_1 & c_2 & - & - & c_{n-1} \\ c_{n-1} & c_n & c_1 & - & - & c_{n-2} \\ & & & & & \\ c_2 & c_3 & - & - & c_n & c_1 \end{bmatrix}$$

is called a circulant. Each row is just the previous row circled forward one step, so that the entries in each rows are just a cyclic permutation of those of the first. Circulant matrices have nice structures which are exploited to make preconditioners.

The block Toeplitz matrice is a matix $T \in R^{MNXMN}$ partitioned as

$$T=\begin{pmatrix} T_{1,1} & . & . & . & . & . & . & T_{i,M} \\ & & & & & & & \\ T_{M,1} & . & . & . & . & . & . & T_{M,M} \end{pmatrix}$$

and such that

$\forall$ i, j = 1, . . . , M: $T_{ij} = T_{i\text{-}j}$ and $T_m = (t_m, h - k)^N$ h, k = 1

whence we can write

$$T=\begin{bmatrix} T_0 & T_{-1} & . & . & . & . & . & T_{-M+1} \\ T_1 & T_0 & & & & & & \\ & & & & & & & \\ & & & & & & & \\ T_{M-1} & & & & & T_1 & T_0 & \end{bmatrix}, T_m=\begin{bmatrix} t_{m,0} & t_{m,-1} & . & . & . & . & . & t_{m,N-1} \\ t_{m,1} & t_{m,0} & & & & & & \\ \vdots & & \vdots & & & & & \\ \vdots & & & \vdots & & & & \\ \vdots & & & & \vdots & & & \\ \vdots & & & & & & & \\ & & & & & & t_{m,1} & t_{m,0} \\ t_{m,N-1} & . & . & . & . & . & & \end{bmatrix}$$

From model problem, equation (2.8) which is a type of Toeplitz block matrix and is written in the form

$$T=\begin{bmatrix} T_0 & I_N & & & & & & \\ I_N & T_0 & T_N & & & & & \\ \vdots & \vdots & \vdots & & & & & \\ & \vdots & \vdots & \vdots & & & & \\ & & \vdots & \vdots & \vdots & & & \\ & & & \vdots & \vdots & \vdots & & \\ & & & & \vdots & \vdots & \vdots & \\ & & & & & \vdots & \vdots & \vdots \\ & & & & & & \vdots & \vdots & I_N \\ & & & & & & & \vdots & T_0 \end{bmatrix} \qquad 3.1$$

I_N is the N x N identity matrix and for n = 5

$$T_0=\begin{bmatrix} 6 & -1 & 0 & 0 & 0 \\ -1 & 6 & -1 & 0 & 0 \\ 0 & -1 & 6 & -1 & 0 \\ 0 & 0 & -1 & 6 & -1 \\ 0 & 0 & 0 & -1 & 6 \end{bmatrix}$$

T can be written as

$$T=I_M \otimes (61-W_N) - W_N \otimes I_N \qquad (3.2)$$

Where

$$W_N = \begin{bmatrix} 0 & 1 & & & & & & & & \\ 1 & 0 & 1 & & & & & & & \\ & 1 & 0 & 1 & & & & & & \\ & & 1 & 0 & 1 & & & & & \\ & & & \vdots & \vdots & \vdots & & & & \\ & & & & \vdots & \vdots & \vdots & & & \\ & & & & & \vdots & \vdots & \vdots & & \\ & & & & & & \vdots & \vdots & \vdots & \\ & & & & & & & \vdots & \vdots & 1 \\ & & & & & & & & 1 & 0 \end{bmatrix} \quad 3.3$$

and ⊗ is the kronecker product

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1M}B \\ \vdots & & \\ \vdots & & \\ \vdots & & \\ \vdots & & \\ a_{M1}B & \cdots & a_{MM}B \end{bmatrix}$$

Now, we construct the circulant preconditioner S_N for T by preserving N consecutive diagonals in T and using them to form a circulant matrix. We note that our preconditioner is block version of the point – circulant T_0.

$$S_N = \begin{bmatrix} T_0 & I_N & & & & & & I_N \\ I_N & T_0 & T_N & & & & & 0 \\ 0 & \vdots & \vdots & \vdots & & & & \\ & \vdots & \vdots & \vdots & & & & \\ & & \vdots & \vdots & \vdots & & & \\ & & & \vdots & \vdots & \vdots & & \\ & & & & \vdots & \vdots & \vdots & \\ & & & & & \vdots & \vdots & \vdots \\ & & & & & \vdots & \vdots & \vdots \; I_N \\ I_N & & & & & I_N & \vdots & T_0 \end{bmatrix} \quad 3.4$$

$$S_N^{-1}T = \frac{6929}{6724}\begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{pmatrix} \qquad \textbf{3.5}$$

COMPUTATIONAL RESULTS.

Table 3.1

No. of iteration for the convergence of the 3- dimensional Poisson equation (2.1) using circular preconditioner.

	Number of iteration
N = 3: 27*27 Matrix	1
N = 4: 64*64Matrix	2
N = 5: 125*125Matrix	2
N = 6: 216*216 Matrix	2
etc.	

4. CONCLUDING REMARKS

In this work, we have successfully carried out computational exposition on large systems using the Preconditioned Conjugate Gradient method. Result of our work attest to the success of the method. The condition number of the preconditioned matrix is 1.03. Thus the eigenvalues are clustered in which case solution is expected in 1 iteration however, due to rounding errors or discretization errors, the system converged in 2 iterations in all cases.

REFERENCES

[1] Golub, Gene and Ortega, James M. (1993), "Scientific Computing An Introduction with parallel computing", Academic Press.

[2] Jennings, A. (1979), The Solution of Sparse Linear, Equation by the Conjugate Gradient Method, International journal for Numerical method in Engineering.

[3] Kershaw, D. (1978), The Incomplete Choleski – Conjugate Gradient Method for the Iterative Solution of Systems of Linear Equation. J. Comp. Phys. Vol. 26: pp. 43 – 65.

[4] Ketter, L.R. and Prawel, P.S. (1979), "Modern Methods of engineering computation", McGraw – Hill book company.

[5] Miejerink, J.A. and Vander Vorst, H. (1977), An Iterative Solution Method for Linear System of which the coefficient matrix is a symmetric M – matrix math. Comp., Vol. 31: pp. 148 – 162.

[6] Olkin, J. (1986), Linear and non Linear Deconvolution problem. Ph.D Thesis, Rice University, Houston, Texas.

[7] Stewart, S.W. (1973), "Introduction to matrix computations" Academic Press, New York.

[8] Strang, G. (1986), A proposal for Toeplitz matrix calculatiuon, Stud. Appl. Math. Vol. 74: pp. 171 – 176.