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OSCILLATORY AND ASYMPTOTIC PROPERTIES OF ODD ORDER DIFFERENCE EQUATIONS

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ABSTRACT

The oscillation criteria for certain odd order difference equations are established. Indeed, we generalise the results of Smith [4] on the oscillatory and asymptotic behaviour for certain third order difference equations to a class of general odd order difference equations.

INTRODUCTION

Smith [4] studied the oscillatory and asymptotic behaviour of third order difference equations of the form

$$\Delta^3 U_n - P_n U_{n+2} = 0 \tag{1}$$

Here Δ denotes the differencing operation

$$\Delta U_n = U_{n+1} - U_n \quad \text{for each } n \geq 1 \tag{2}$$

The results of Smith [4] are the discrete analogue of the results of Taylor [5] which are the e extensions of the work of Hanan [1]. The method is concerned with a characterization of the existence of oscillatory solutions of (1) in terms of the behaviour of non-oscillatory solutions.

In the present study equation (1) is generalised to the form

$$\Delta^3 U_{n+m-3} - P_{n+m-3} U_{n+m-1} = 0 \tag{2}$$

We show that (2) has both oscillatory and non-oscillatory solutions for $P_{n+m-3} > 0$ for in any odd number greater than or equal to three.

Definition (1)

By the graph of a solution $\{U_n\}$ of (2) we mean the polygonal path connecting the points (n, U_n), $n \geq 1$. Any point where the graph $U = \{U_n\}$ intersects the real axis is called a node.

Definition (2)

A solution of equation (2) will be called oscillatory if it has arbitrary large nodes; otherwise, it is said to be non-oscillatory.

Lemma 1

If $U = \{U_n\}$ is a solution of equation (2) satisfying

$$U_k \geq 0, \Delta U_k \geq 0, \ldots\ldots, \Delta^{m-1} U_k > 0$$

for some choice of $k \geq 1$, then

$$U_k \geq 0, \Delta U_k \geq 0, \ldots\ldots, \Delta^{m-1} U_k > 0$$

for some choice of $k \geq 1$, then

$$U_n > 0, \Delta U_n > 0, \ldots., \Delta^{m-1} U_n > 0$$

for each $n \geq k + 2$

Proof

We show that the lemma is true for $n = k+2$.

Note that

$$\Delta\left(\Delta^2 U_{k+m-1}\right) = P_{k+m-1} U_{k+m+1} > 0$$

Thus

$$\Delta^2 U_{k+m} \geq \Delta^2 U_{k+m-1} \quad \text{and}$$

We have

$$\Delta^2 U_{k+m-1} > 0$$

Similarly, $\Delta^2 U_{k+m-1} > 0$ implies that $\Delta U_{k+m-1} > 0$ which in turn implies $U_{k+m-1} > 0$. Hence the result holds for each $n > k + 1$ proves the lemma.

Remark

The above result shows that equation (2) always has non-oscillatory solutions. Furthermore, the positivity of the sequence coefficients $\{P_{n+m-3}\}$ places rather stronger restrictions on the behaviour of the non-oscillatory solutions of equation (2).

Theorem 1

Let $U = \{U_n\}$ be a non-oscillatory solution of equation (2).

Then for all sufficiently large n

$$U_n \Delta_n \Delta^2 U_n \Delta^3 U_n \ldots\ldots \Delta^{m-1} U_n \neq 0$$

and either

$$U_n > 0, \Delta U_n > 0, \Delta^2 U_n > 0, \ldots\ldots, \Delta^{m-1} U_n > 0 \tag{3}$$

or

$$U_n > 0, \Delta U_n > 0, \ldots\ldots, \Delta^{m-2} U_n > 0, \Delta^{m-1} U_n < 0 \tag{4}$$

Proof

Assume that $U = \{U_n\}$ is a non-oscillatory solution of (2), where $U_n > 0$ for each $n \geq N$. Form $m = 3$, equation (1) becomes

$$\Delta^3 U_n = \Delta\left(\Delta^2 U_n\right) = P_n U_{n+2} > 0$$

for all $n \geq N$, hence $\Delta^2 U_n$ is increasing and eventually of one sign. So it follows that M exists, $M \geq N$ for ΔU_n and $\Delta^2 U_n$ are sign definite for all $n \geq M$. Hence

$$U_n \Delta U_n \Delta^2 U_n \neq 0$$

for every $n \geq M$

For $m = 5$ equation (2) becomes

$$\Delta^3(U_{n+2}) = \Delta(\Delta^2 U_{n+2}) = P_{n+2}U_{n+4} > 0$$

Hence $\Delta^2 U_{n+2}$ is increasing and eventually of one sign.
So, it follows that M exists, $M \geq N$
From equation (2), we have

$$\Delta^3 U_{n+k-1} = \Delta(\Delta^2 U_{n+k-1}) = \Delta(\Delta^2 U_{n+k-3} + 2\Delta U_{n+k-3} + U_{n+k-3}) > 0$$

hence $\Delta^2 U_{n+k-1}$ is increasing and eventually of one sign.
Since it is true for $m = k + 2$ it is true for all k.
Clearly

$$\Delta^3(U_{n+m-3}) = \Delta(\Delta^2 U_{n+m-3}) = P_{n+m-3}U_{n+m-3} > 0$$

hence $\Delta^2 U_{n+m-3}$ is increasing and eventually of one sign.
So it follows that M exists, $M \geq N$ for which U_n, $\Delta^2 U_n, \ldots, \Delta^{m-2}U_n$ and $\Delta^{m-1}U_n$ are sign definite for all $n \geq M$.
Hence

$$U_n \Delta U_n \Delta^2 U_n \Delta^3 U_n \ldots\ldots \Delta^{m-1} U_n \neq 0$$

for every $n \geq M$.
The cases

$$U_n > 0, \Delta U_n < 0, \Delta^2 U_n > 0, \Delta^3 U_n < 0 \ldots\ldots\ldots$$

$$\Delta^{m-2} U_n < 0 \quad \Delta^{m-1} U_n > 0 \quad n \geq M.$$

and

$$U_n > 0, \Delta U_n > 0, \Delta^2 U_n \Delta^3 U_n > 0, \quad \Delta^3 U_n > 0 \ldots\ldots \Delta^{m-1} U_n < 0 \quad n \geq 0$$

are clearly impossible since

$$\Delta^i U_n \quad \Delta^{i+1} U_n > 0 \quad \text{for all n}$$

Sufficiently large implies

$$\text{Sgn } \Delta^{i+1} U_n = \text{Sgn } \Delta^i U_n \quad i > 1$$

eventually and the proof is complete.
Define by S^* the n-dimensional vector space of solutions of equation (2) for each U E S^* define

$$G_n^m = G^m[U_n] = (\Delta U_{n+m-3})^2 - 2U_{n+m-2}\Delta^2 U_{n+m-3} \quad (5)$$

Lemma
If $U = \{U_n\}$ is a solution of equation (2), then the functional defined by (5) is decreasing.

Proof
From (5), we have

$$G^m[U_n] = (\Delta U_{n+m-3})^2 - 2U_{n+m-2}\Delta^2 U_{n+m-3}$$

Taking the difference, it is easy to see that

$$\Delta G_n^m = \left(\Delta^2 U_{n+m-3}\right)^2 - 2P_{n+m-2}U^2_{n+m-1} \qquad (6)$$

The proof is complete

Remark
There can exist at most one value of n such that $U_n = U_{n+1} = 0$

Theorem 2
There exists U E S^- satisfying $G^m{}_n > 0$ for each $n \geq 1$

Proof
Let $x^1, x^2, \ldots x^m$ be a basis for S^-.
For every positive integer, K, define

$$U_n^k = A_1^k x^1 + A_2^k x^2 + \ldots A_m^k x^x$$

where $A^k{}_1$ are chosen in a way that

$$U_{k+1}^k = U_{k+2}^k = 0 \quad \text{and}$$

Let $U_{k+1}^k > 0$. It follows from lemma (2) that $G^m\left[U_n^k\right] > 0$ for all $1 \leq n \leq k$.

Put $A_k = \left(A_1^k \ldots\ldots A_m^k\right)$ where A_i^k are as defined earlier on. Then $\| A_k \| = 1$ for each k.
Due to compactness of the unit ball in IR^m, it follows that the sequence $\{A_k\}$ has a convergent subsequence $\{A_{ki}\}$ such that $A_{ki} \to A = (A_1, \ldots A_m)$ as $i \to \infty$
Where

$$\sum_{i=1}^m (A_i)^2 = 1$$

Let U be defined by

$$U_n = A_1 x^1 + A_2 x^2 + \ldots + A_m x^m$$

Then clearly it is a nontrivial solution of equation (2).
Now $G^m[U_n] > 0$ for all $n \geq 1$, for if not there is an integer j such that $G^m[U_j] < 0$, since $U_j^{ki} \to U_j$. We can infer that $G^m\left[U_j^{ki}\right] \to G^m\left[U_j\right] < 0$.

Choose a positive integer M such that for all $i > M, G^m\left[U_j^{ki}\right] < 0$ and $k_i < j$.

Since $G^m[U_n]$ is decreasing and $G^m\left[U_{ki}^{ki}\right] > 0$, we have for $i > M$

$$0 < G^m\left(U_{ki}^{ki}\right) < G^m\left[U_j^{ki}\right] < 0$$

From this contradiction, we see that $G^m[U_n] > 0$ for each n.
This complete the proof.
We now introduce a quasi-adjoint difference equation

$$\Delta^3 V_{n+m-1} + P_{n+m-4} V_{n+m-2} = 0 \qquad (6)$$

$m \geq 3$ is any odd number

The adjoint of equation (1) is defined as

$$\Delta^3 \propto_{n+m-3} + P_{n+m-2} \propto_{n+m-2} = 0 \qquad (7)$$

M ≥ 3 is any odd number

The remainder of this section contains results which show that the solutions of the difference equation (2) and (6) satisfy relations similar to those that exist between two adjoint differential equations.

For this reason, we say equation (2), and (6), are quasi-adjoint.

Turning to equation (6), we show that it always has non-oscillatory solutions. A proof based on the following lemma is given.

Lemma (2)

If $v = \{V_r\}$ is a solution of equation (6) satisfying

$$V_r \geq 0, \Delta V_k > 0, \Delta^2 V_k < 0 \;\; \Delta^3 V_k > 0 \ldots\ldots \Delta^{m-2} V_k < 0, \Delta^{m-1} V_k > 0$$

for some integer $r > k \geq 1$, then

$$V_r > 0, \Delta V_k > 0, \Delta^2 V_k < 0 \;\; \Delta^3 V_k > 0 \ldots\ldots \Delta^{m-2} V_k < 0, \Delta^{m-1} V_k > 0$$

for each $1 \geq k < r$

Proof

We show the lemma is true for $k = r-1$

Note that

$$\Delta\left(\Delta^2 V_{n+r-4}\right) = -P_{n+r-4} V_{n+r-2} \leq 0$$

Thus

$$\Delta^2 V_{n+r-3} \leq \Delta^2 V_{n+r-4} \quad \text{and}$$

We have that

$$\Delta^2 V_{n+r-4} > 0$$

Similarly

$$\Delta^2 V_{n+r-4} > 0$$

$$\rightarrow \Delta^2 V_{n+r-4} < 0$$

which in turn implies that $V_{n+r-4} > 0$.

Proceeding this way we have $V_{n+r-4} > 0$.

Hence the result holds for $k = r-1$,

Repeating this process for each $1 \leq k \leq r-1$ proves the lemma.

Theorem 3

Let $k \geq 1$. There exists a solution

$V = \{V_n\}$ of (6) satisfying

$$V_n > 0, \Delta V_n < 0, \Delta^2 V_n > 0, \Delta^3 V_n < 0, \Delta^{m-1} V_n > 0 \qquad (8)$$

for each $n \geq N \geq r$

Proof

Let $\propto_1, \propto_2, \propto_3, \ldots, \propto_m$ be a basis for S^+ the solution space of equation (6). For each positive integer C, define

$$V_n^c = B_1^c \propto_1 + B_2^c \propto_2 \ldots\ldots + B_m^c \propto_m$$

where $B^c{}_1$ are chosen in such a way that

$$V_c^c = V_{c+1}^c = 0 \quad \text{and}$$

$$\sum_{i=1}^{m} \left(B_i^c\right)^2 = 1$$

Assuming $V_{c+2}^c > 0$ and proceeding as in the proof of theorem (2), we can find a sequence $\{C_I\}$ of positive integers such that

$$\underset{Ci \to \infty}{\text{Limit}} \; V_n^{ci} = V_n$$

Defines a non-trivial solution of equation (6).

We see, by lemma (4) that

$$V_n \geq 0, \Delta V_n \leq 0, \ldots\ldots, \Delta^{m-1} V_n > 0$$

$$\Delta^3 V_{n+m-3} = -P_{n+m-4} V_{n+m-2} \leq 0$$

for all n.

If $V_{n0} = 0$ for $n_0 \geq r$ then since V_n is non-increasing $V_n = 0$ for all $n \geq n_0$. Contradicting the fact V is non-trivial. Hence $M_0 \geq V$ exists such that $V_n > 0$ for every n $> M_0$ in which case

$$\Delta^3 V_{n+m-3} = -P_{n+m-4} V_{n+m-2} < 0$$

for all $n > M_0$. It then follows from another application of lemma (3) that $V_n\, \Delta V_n\; \Delta^2 V_n, \Delta^{m-1} V_n \neq 0$ for all $n \geq 1$ and furthermore and furthermore

$$V_n > 0, \Delta V_n < 0, \Delta^2 V_n > 0, \ldots\ldots, \Delta^{m-1} V_n > 0 \text{ for each n}$$

This completes the proof.

Following Taylor [5] we term solutions of equation (2), which satisfy equation (2) as strongly increasing and those which satisfy (3), as minimally increasing. Those solutions of equation (6) which satisfy equation (8) we term as strongly decreasing.

Oscillation Properties of Equation (2)

In this part, we examine the asymptotic behaviour of certain solutions of equation (2), we will also consider some general relationships that exists between the solutions of equation (2) and those of equation (6). In the event that equation (2) has oscillatory solutions, our main result will show that even stronger restrictions are placed on the non-oscillatory solutions of equations (2) than required by theorem (1). In fact, we will show that minimally increasing solutions cannot be introduced into the solution space without forcing out all of the oscillatory solutions.

Theorem 4

Let $U = \{U_n\}$ be a solution of equation (7) satisfying $G^m{}_n > 0$ for each $n \geq 1$ and $m \geq 3$ fixed odd number, then

(i) $\sum_{n=1}^{\infty}\left(\Delta^2 U_{n+m-3}\right)^2 < \infty$ and

(ii) $\sum_{n=1}^{\infty} P_{n+m-3} U^2_{n+m-1} < \infty$

Proof

Since $G^m{}_n > 0$ for each $n \geq 1$, differencing $G^m{}_r$ and summing from 1 to (r-1) yields

$$0 < G_r^m = G_1^m - \sum_{j=1}^{r-1}\left(\Delta^2 U_{j+m-3}\right)^2 - 2\sum_{j=1}^{r-1} P_{j+m-3} U_{j+m-1} 2$$

Thus

$$\sum_{j=1}^{r-1}\left(\Delta^2 U_{j+m-3}\right)^2 + 2\sum_{j=1}^{r-1} P_{j+m-3} U^2_{j+m-1} < G_1^m$$

letting $r \to \infty$ establishes each of (i) and (ii), since $G^m{}_1$ is dependent of r.

Corollary

Suppose

Limit inf $P_{n+m-3} > 0$

$n \to \infty$

If $U = \{U_n\}$ is a solution of equation (2) satisfying $G^m{}_n > 0$ for fixed $m \geq 3$ any odd and each n then

$$\sum_{n=1}^{\infty} U^2_{n+m-3} < \infty$$

we can now exhibit the discrete Lagrange bilinear concomitant for solutions of equation (2), and (6) for $(U,V) \in S^- \times S^+$ define

$$F_n^m = F^m\left[U_n V_n\right] = U_{n+m-2}\Delta^2 V_{n+m-2} - \Delta U_{n+m-2}\Delta U_{n+m-2} + V_{n+m-2}\Delta^2 U_{n+m-2} \qquad (9)$$

for $m \geq 3$ any odd number and $n \geq 1$

Theorem 5

If $U \in S^-$ and $V \in S^+$ then the function defined by (9) is a constant that is determined by the initial values of U and V.

Proof

Differencing (9), we obtain

$$\Delta F_n^m = \Delta^2 V_{n+m-2}\ \Delta U_{n+m-2} - \Delta U_{n+m-2}\ \Delta^2 V_{n+m-2} - \Delta U_{n+m-2}\ \Delta V_{n+m-2} + \Delta V_{n+m-2}\ \Delta U_{n+m-2}$$

Using the fact that

$$\Delta V_{n+m-1} = \Delta^2 V_{n+m-2} + \Delta V_{n+m-2}$$

Therefore

$$\Delta F_n^m = 0$$

The proof is complete.

Let $x_1, x_2, x_3, \ldots\ldots x_{m-1}$ be independent solution of (1), the wronskian

$$W_n^m = w^m(x_1, x_2 x_3, \ldots\ldots x_{m-1})$$

$$= \begin{vmatrix} x_1 & x_2 & \cdots\cdots & x_{m-1} \\ \Delta x_1 & \Delta x_2 & \cdots\cdots & \Delta x_{m-1} \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \Delta^{m-2} x_1 & \Delta^{m-2} x_2 & \cdots\cdots & \Delta^{m-2} x_{m-1} \end{vmatrix}$$

It is known that w_n^m is a nontrivial solution of (6).

Moreover, if $x_1, x_2, \ldots\ldots x_{m-1}$ do enjoy the same oscillatory character, then $W = \{W^m{}_n\}$ is a non-trivial solution of (6).

Similarly, if $y_1, y_2, \ldots y_{m-1}$ are solutions of equation (6) that are of distinct oscillatory nature then $w(y_1, y_2, \ldots y_{m-1})$ is a non-trivial oscillatory solution of equation (2). We therefore have the following result which is a discrete analogue of Hanan [1]

Corollary 6

Equation (6) is oscillatory if and only if equation (2) is oscillatory.

Let $z_1, z_2, \ldots z_m$ be solutions of equation (1).

Expanding the wronskian

$$R_n^m = R^m[z_1 z_2 \ldots\ldots\ldots, z_m]$$

along its mth column, we obtain the following relationship between

F_n^m, w_n^m and R_n^m

$$R_n^m = F^m[w(z_1, z_2 \ldots\ldots, z_{m-1}), z_m]$$

Theorem 7

If V is non-oscillatory solution of equation (6) then (m-1) independent solutions of equation (1) satisfy the self-adjoint (m-1) the order difference equation

$$\Delta\left(\frac{\Delta U_{n+m-2}}{V_{n+m-2}}\right) + \left(\frac{\Delta^2 V_{n+m-2}}{V_{n+m-2} V_{n=m-1}}\right) U_{n+m-2} = 0 \qquad (10)$$

Proof

Since v is fixed in S^+, we have

$$K_F = \left\{ U \in S^{-1} \middle| F^m\left[U_n.V_n\right] = 0 \right\}$$

is the kernel of the linear functional

$$F_n^m : S^- \to IR$$

where IR denotes the set of real numbers. If $V_n > 0$ $n \geq N$ then

$$x \in K_F \to$$

$$\Delta\left(\frac{\Delta x_{n+m-3}}{V_{n+m-2}}\right) + \left(\frac{\Delta^2 V_{n+m-2}}{V_{n+m-2}\ V_{n+m-1}}\right) \quad x_{n+m-2} = 0$$

The result follows, since

$$\dim K_F + \dim I\hat{R} = \dim S^-$$

We now derive an oscillation condition for equation (2) in terms of equation (9)

Theorem 8

The following two statements are:

(1) Equation (2) is oscillatory

(2) Equation (10) is oscillatory

Proof

Suppose that condition (i) holds then by theorem (6) equation (6) is oscillatory.

Let x_1 be an oscillatory solution of (6).

Consider

R^m [x_1, x_2,.....x_2] [x_2 occurring (m-1) times]

Where x_2 is oscillatory solution of (6) whose existence was shown in theorem 3.

Thus

$$F^m\left[w^m\left(x_1, x_2, \ldots\ldots, x_2\right), x_2\right] = 0 \qquad \text{(m – 2) times}$$

and we find that

$W^m\left(x_1, x_2, \ldots\ldots x_2\right)$ is an oscillatory (m – 2) times

solution of

$$\Delta\left(\frac{\Delta w_{n+m-4}}{V_{n+m-3}}\right) + \left(\frac{\Delta^2 V_{n+m-3}}{V_{n+m-3}\ V_{n+m-2}}\right) \quad w_{n+m-3} = 0$$

This proves the first part of the theorem. Suppose that condition (ii) holds where V is a non-oscillatory solution of equation (6) with $V_n > 0$ $n \geq N$.

If U is an oscillatory solution of equation (9) then $U \in K_F$ and in particular $U \in S^-$.

This completes the proof.

Remark
Since the nodes of linearly independent solutions of (10) separate each other and those of linearly dependent solutions coincides, it follows that either all solutions of "(10) are oscillatory or all solutions of (10) are non-oscillatory. (See McCarthy [13]).
We therefore have the following corollary of theorem 8.

Corollary 2
If equation (2) is oscillatory then S^- has a basis consisting of one non-oscillatory solution and (m-1) oscillatory solutions.

Theorem 9
The following two statements are equivalent.

(i) Equation (2) is oscillatory

(ii) For every non-oscillatory solution U of equation (2) there exists an integer N for which U of equation (2) there exists an integer N for which
$U_n > 0,\ \Delta U_n > 0,\ \Delta^2 U_n > 0,\ \ldots,\ \Delta^{m-1} U_n > 0\ \ n \geq N$

Proof
Suppose that condition (1) holds and equation (1) has a solution satisfying

$$y_n > 0,\ \Delta y_n > 0,\ \ \Delta^2 y_n > 0,\ \ldots,\ \Delta^{m-2} y_n > 0,\ \Delta^{m-1} y_n < 0\ \ n \geq N$$

By corollary (6) and the latest remark, there exist (m-1) independent oscillatory solutions of equation (2), every linear combination of which is oscillatory.
Let $f^1, f^2, \ldots, f^{m-1}$ be such pair of solutions with

$$f_N^1 = 0, f_N^2 \neq 0, f_N^2 \neq 0, f_N^3 \neq 0, \ldots, f_N^{m-1} \neq 0.$$

Let

$$\phi_0 = y_n - \sum_{i=2}^{m-1} di\ f_n^i$$

where di are constants chosen in such a way that $\phi_N = 0$.
Consider

$$W^m\left(\phi_n, f_n^i\right) \qquad 1 \leq i \leq m-1$$

Now

$$W^m\left(\phi_N, f_N^i\right) = 0$$

here there exists constants C_1, C_2,
with

$$C_1^2 + C_2^2 \neq 0$$

such that

$$C_1 \phi_N + C_2 f_N^1 = 0$$

$$C_1\Delta\phi_N + C_2\Delta f_N^1 = 0$$

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$$C_1\Delta^{m-2}\phi_N + C_2\Delta f_N^{m-1} = 0$$

put

$$U_n = C_1\,\phi_N + C_2\,f_n$$

Then U has a double zero at N and

$$U_n = c_1 y_n + \Psi_n$$

where

$$\Psi_n = C_2 f_n^1 - \sum_{i=2}^{m-1} c_2 \text{ dif}_n^i$$

is an oscillatory solution of (2). Since U is non-trivial, we may suppose without loss of generality that

$$\Delta^2 U_{n+m-3} > 0$$

As a consequence of lemma (1)

$$\lim_{n\to\infty} \Delta^2 U_{n+m-3} = \infty$$

Moreover, the relations

$$y_n > 0, \Delta^2 y_n < 0, \Delta^3 y_n > 0, \ldots\ldots, \Delta^4 y_n < 0, \ldots.. \Delta^{m-1} y_n < 0\ \Delta^m y_n > 0\ n \geq N$$

imply that

$\{\Delta y_n^w\}$ is asymptotic to a finite constant

Now

$T(q, \Delta u_{n+m-3}) = (\Delta^3 U_{n+m-3})(9 - n) + \Delta u_{n+m-3} \quad n \leq q \leq n+1, \quad n \leq 1,$ defines the graph of $\{U_n\}$

Let $\{q_1\}$ be an increasing sequence of nodes of $\{\Delta\Psi n\}$.

Then at each q_i we have

$$T(q_i, \Delta u_{n+m-3}) = C_1 T(q_i, + \Delta y_{n+m-3}) \tag{11}$$

We have arrived at a contradiction since the left member of (ii) becomes unbounded as i $\to \infty$. This contradiction proves the first part of the theorem.

Suppose that condition (ii) holds and every oscillatory solution of equation (2) I strongly increasing. if u is a non-oscillatory solution of equation (2) such that the condition (i) holds for each n ≥ N.

Then differencing $G^m n$, we obtain as a result of (4) the inequality

$$G_n^m \leq -(\Delta^2 U_{n+m-3})^2 \tag{12}$$

Summing both sides of (12) from N to k-1, we obtain

$$G_r^m \le G_n^m - \sum_N^{r-1}\left(\Delta^2 U_{n+m-3}\right)^2$$

$$\le G_n^m - \left(\Delta^2 U_{n+m-3}\right)^2 \sum_N^{r-1} 1 \to \infty$$

as $r \to \infty$.

Hence

$$\underset{n\to\infty}{\text{Limit}}\ G_n^m = \infty$$

holds for every non-oscillatory solution of equation (2).

By theorem 2, there exists a solution of equation (1) that satisfies $G^m{}_n > 0$ for each $n \ge 1$ such a solution clearly satisfies

$$\underset{n\to\infty}{\text{Limit}}\ G_n^m \ge 0$$

and hence must be oscillatory.

We record as our final result a sufficient condition for equation (2) to be oscillatory in terms of the coefficient function $\{P_{n+m-3}\}$. This result is a discrete analogue of Jones [2].

Theorem 10

If

$$\sum_{n=1}^{\infty} P_{n+m-3} = -\infty$$

then equation has oscillatory solution

Proof

In the light of theorem 2, it is enough to show that

$$\underset{n\to\infty}{\mathit{Limit}}\ G_n^m = -\infty$$

for every non-oscillatory solutions of equation (2).

However, this clearly is the case of

$$\sum_{n=1}^{\infty} P_{n+m-3} = \infty$$

for conditions of theorems (2) and (3) imply

$$G_r^m \le G_N^m - 2U^2_{N+m-1} \sum_N^{r-1} P_{n+m-1} \to \infty$$

as $r \to \infty$

where U is a non-oscillatory solution of (2). Hence the proof.

Example
Consider the fifth order difference equation

$$\Delta^3 U_{n+2} - \frac{U_{n+4}}{2(2^{n+4} - 1)} = 0$$

The solution $U_n = 1\text{-}2^{-n}$ is a minimally increasing solution.
As a consequence of theorem 9, every solution is non-oscillatory.

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