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ON THE SUPERPOSITION THEORY OF THE NEAR EARTH'S SURFACE LONGITUDINAL AND TRANSVERSE ELASTIC WAVES.

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ABSTRACT:

The seismic events associated with the superposition of the longitudinal and transverse waves propagating near the Earth's surface are presented. The geophysical manifestations of these developments are usually in the form of the Rayleigh waves and kindred events. This study, consequently, is aimed at analyzing some of the propagation characteristics of the Rayleigh elastic surface waves.

In particular, the wave slowness for the elastic surface waves are analyzed, thereby illustrating the geophysical role of the Rayleigh surface wave slowness. Further, the non-trivial dependence of the phase and group velocities maxima of the surface waves on the material rigidity is shown to be of higher order than those of the previous investigations (Bullen & Bolt, 1985).

1. INTRODUCTION

The process culminating in superposition of the longitudinal and transverse elastic waves near the Earth's surface is a concept well established in the subject of geodynamics. This was initially formulated following the Rayleigh's theory of waves on the boundary of a homogeneous elastic half space (Rayleigh, 1885). Consequently, the geophysical phenomena resulting from the interplay of the reflected longitudinal and transverse waves propagating in the condition of total inner reflection constitute a special type of vibration called Rayleigh surface waves. Unlike water waves, Rayleigh waves are sustained by the interactions between the inertia and elasticity forces in the solid medium (Savarensky, 1975). Further, away from the generating source, these waves develop into a pattern of inhomogeneous plane surface waves with the amplitude diminishing below the Earth's surface.

The longitudinal and transverse waves respectively propagate with the speeds α and β where $\alpha > \beta$. Nevertheless, the vibrations generated by the superposition of the two propagate with a common speed c_R which is less in magnitude than either β or α. c_R is then the speed of the Rayleigh surface wave made; this is unlike the evolution of the body wave with propagation speed greater than α and β respectively.

Besides the observed amplitude weakening with the increasing depth below the Earth's surface associated with the evolutions of the progressive Rayleigh's waves, these waves induce orbital motion on the particles of the solid medium through which they pass. Geometrically, the orbit is incidentally elliptic. In addition, these developments give rise to guided wave phenomena which are trapped near the Earth's surface and therefore, accumulate sufficient energy to propagate over a long distance away from the generating source. We nov re-examine some of the inherent properties of these trapped progressive modes.

2. THE RAYLEIGH'S WAVE SLOWNESS

The following parameters K_α and K_β related to the longitudinal and transverse elastic waves are respectively defined as follows:

$$K_\alpha^2 = \left(K^2 - \frac{\omega^2}{\alpha^2}\right) \qquad \ldots (2.0)$$

$$K_\beta^2 = \left(K^2 - \frac{\omega^2}{\beta^2}\right) \qquad \ldots (2.1)$$

The Rayleigh's function $R(K,\omega)$ for the wave frequency ω and wave number K is defined as follows:

$$R(K,\omega) = \left(2K^2 - \frac{\omega^2}{\beta^2}\right)\left(2K^2 - \frac{\omega^2}{\alpha^2}\right) - K^2\left(K^2 - \frac{\omega^2}{\alpha^2}\right)^{1/2}\left(K^2 - \frac{\omega^2}{\beta^2}\right)^{1/2} \qquad \ldots (2.2)$$

$$= \omega^4\left(\frac{2K^2}{\omega^2} - \frac{1}{\beta^2}\right)\left(\frac{2K^2}{\omega^2} - \frac{1}{\alpha^2}\right) - K^2\omega^2\left(\frac{K^2}{\omega^2} - \frac{1}{\alpha^2}\right)^{1/2}\left(\frac{K^2}{\omega^2} - \frac{1}{\beta^2}\right)^{1/2}$$

Since the wave speed, $c = \frac{\omega}{K}$, we define the following wave slowness as in the form

$\gamma = \frac{1}{c}$, $\gamma_\beta = \frac{1}{\beta^2}$, $\gamma_\alpha = \frac{1}{\alpha^2}$. c is now interpreted as the apparent wave speed along the boundary. Equat n (2.2) can now be expressed in terms of slowness parameters as follows:

$$R(\gamma^2) = \left(2\gamma^2 - \gamma_\beta^2\right)\left(2\gamma^2 - \gamma_\alpha^2\right) - \gamma^2\left(\gamma^2 - \gamma_\beta^2\right)^{1/2}\left(\gamma^2 - \gamma_\alpha^2\right)^{1/2} \qquad \ldots (2.3)$$

The expressions $\lim_{\gamma \to \gamma_\beta} R(\gamma) = \lim_{\gamma \to \gamma_\alpha} R(\gamma) = 0$ are not realizable physically for in the case of the surface seismic waves, $\gamma_\beta > \gamma_\alpha > \gamma$.

Equation (2.3) may be re-arranged in the form

$$R(\gamma)=\gamma^4\left(2-\frac{\gamma_\beta^2}{\gamma^2}\right)\left(2-\frac{\gamma_\alpha^2}{\gamma^2}\right)-\gamma^4\left(1-\frac{\gamma_\beta^2}{\gamma^2}\right)^{1/2}\left(1-\frac{\gamma_\alpha^2}{\gamma^2}\right)^{1/2}$$

$$=\gamma^4\left[4-\frac{2}{\gamma^2}\left(\gamma_\beta^2+\gamma_\alpha^2\right)+\frac{\gamma_\alpha^2\gamma_\beta^2}{\gamma^4}\right]-\gamma^4\left[1-\frac{\gamma_\beta^2}{2\gamma^2}-\frac{1}{8}\frac{\gamma_\beta^4}{2\gamma^2}+\ldots\right]\left[1-\frac{\gamma_\alpha^2}{2\gamma^2}-\frac{1}{8}\frac{\gamma_\alpha^4}{2\gamma^2}+\ldots\right]$$

$$=\gamma^4\left\{\left[4-\frac{2}{\gamma^2}\left(\gamma_\beta^2+\gamma_\alpha^2\right)+\frac{\gamma_\alpha^2\gamma_\beta^2}{\gamma^4}\right]-\left[1-\frac{1}{2\gamma^2}\left(\gamma_\beta^2+\gamma_\alpha^2\right)-\frac{1}{8\gamma^4}\left(\gamma_\beta^4+\gamma_\alpha^4\right)+\frac{\gamma_\beta^2\gamma_\alpha^2}{4\gamma^4}+O\left(\frac{1}{\gamma^6}\right)\right]\right\}$$

$$=\gamma^4\left[4-\frac{2}{\gamma^2}\left(\gamma_\beta^2+\gamma_\alpha^2\right)+\frac{\gamma_\alpha^2\gamma_\beta^2}{\gamma^4}-1+\frac{1}{2\gamma^2}\left(\gamma_\beta^2+\gamma_\alpha^2\right)-\frac{1}{8\gamma^4}\left(\gamma_\beta^2-\gamma_\alpha^2\right)^2+O\left(\frac{1}{\gamma^6}\right)\right]$$

$$=\gamma^4\left[3-\frac{3}{2\gamma^2}\left(\gamma_\beta^2+\gamma_\alpha^2\right)+O\left(\frac{1}{\gamma^4}\right)\right] \qquad \ldots (2.4)$$

Finally,

$$R(\gamma)=\gamma^4\left[3-\frac{3}{2\gamma^2}\left(\gamma_\beta^2+\gamma_\alpha^2\right)+O(1)\right]. \qquad \ldots (2.5)$$

Thus, $\lim_{\gamma\to\infty} R(\gamma)=\infty$ and $R(\gamma)$ is a continuous function of γ with singularity at infinity.

For some $\gamma=\gamma_R={}^{1}\!/_{c_R}$, $R(\gamma_R)=0$ in which $\gamma_R>\gamma_\beta>\gamma_\alpha$. γ_R therefore defines a free surface wave which exists without supporting incident wave. Consequently, the parameters S_α and S_β are now purely complex and are associated with the following elastic wave properties:

$S_\alpha=iS_\alpha'$, $S_\beta=iS_\beta$ where $S_\alpha'=\left(\gamma_R^2-\gamma_\alpha^2\right)^{1/2}$, $S_\beta=\left(\gamma_R^2-\gamma_\beta^2\right)^{1/2}$.

The above representations are, therefore, consistent with the case of the inhomogeneous surface waves trapped near the Earth's free surface. But hypothetically however, these decay exponentially with the increasing depth from the free surface.

3. THE FREQUENCY CHARACTERIZATION OF THE WAVE FORM

Following Asor and Okeke (1998) and Asor (2000), the condition for the existence of the trapped modes of vibrations which are not sustained by the incident waves is given by

$$\left(2K^2-\frac{\omega^2}{\alpha^2}\right)\left(2K^2-\frac{\omega^2}{\beta^2}\right)-4K^2K_\alpha K_\beta=0 \qquad \ldots (3.1)$$

Equation (3.1) can be re-arranged as follows:

$$\omega^4 - 2K^2\omega^2\left(\alpha^2 + \beta^2\right) + 4\alpha^2\beta^2K^2\left(K^2 - K_\alpha K_\beta\right) = 0 \qquad \ldots (3.2)$$

In this section, K_α and K_β are now interpreted as the wave numbers of the longitudinal-P and transverse-SV waves respectively. K is the wave number of the free surface wave arising from the superposition of the P and SV waves.

Equation (3.2) is quadratic in ω^2. If ω_1^2 and ω_2^2 are the two independent roots, we have the following representations:

$$\left(\frac{\omega_1}{K}\right)^2 = \left(\alpha^2 + \beta^2\right) + R_1 \qquad \ldots (3.3)$$

$$\left(\frac{\omega_2^2}{K}\right)^2 = \left(\alpha^2 + \beta^2\right) - R_1 \qquad \ldots (3.4)$$

$$R_1 = \left[\left(\alpha^2 + \beta^2\right)^2 - R_2\right]^{1/2} \qquad \ldots (3.5)$$

$$R_2 = \frac{4\alpha^2\beta^2}{K^2}\left(K^2 - K_\alpha K_\beta\right) \qquad \ldots (3.6)$$

In the case of surface Rayleigh waves with wave number, K_R, $K_R > K_\beta > K_\alpha$ indicating that the wavelength of the pressure wave is longer than that of shear wave which in turn is longer than that of Rayleigh wave in the inhomogeneous elastic material. This definition is confirmed by the observational evidence (Bullen & Bolt, 1985; Okeke & Asor, 2000).

Since the expression given by equation (3.5) is strictly positive, equations (3.3) and (3.4) define two completely decoupled solutions in the frequency spectrum. For a fixed wave period, equation (3.3) gives the lower frequency band whilst equation (3.4) gives the upper frequency band and Fig. 1 depicts the frequency spectral decomposition.

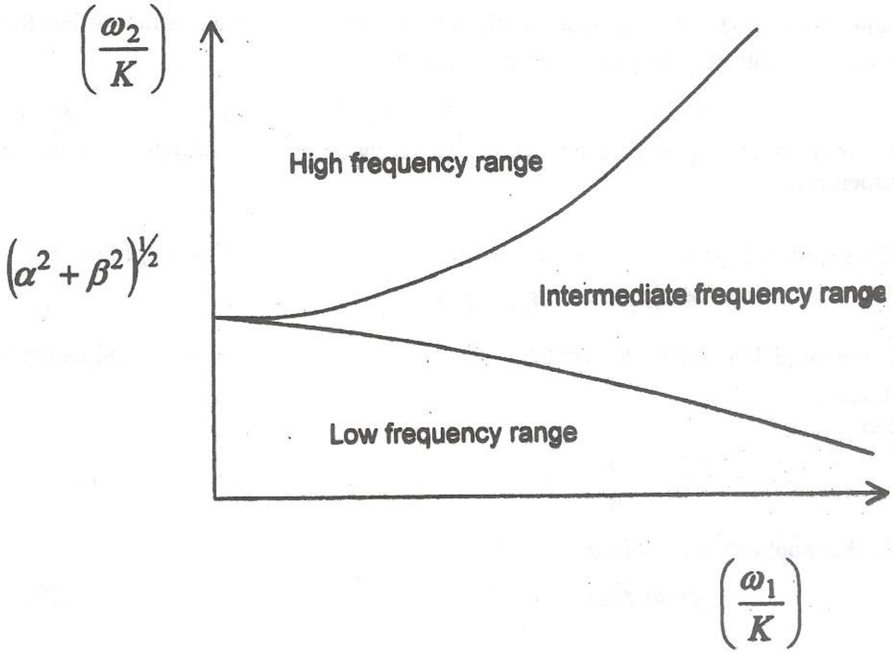

Fig. 1: Partitioned frequency spectrum

Further, equation (3.2) may be re-arranged as follows:

$$f(\omega)=\omega^4-2K^2\omega^2(\alpha^2+\beta^2)+4\alpha^2\beta^2K^2(K^2-K_\beta K_\alpha)$$
$$=[\omega^2-K^2(\alpha^2+\beta^2)]^2+K^2[4\alpha^2\beta^2(K^2-K_\beta K_\alpha)-K^2(\beta^2+\alpha^2)^2] \quad \text{... (3.7)}$$

That is,

$$f(\omega)=[\omega^2-K^2(\alpha^2+\beta^2)]^2+K^2[2\alpha^2\beta^2(2K_\beta K_\alpha+K^2)-K^2(\alpha^4+\beta^4)] \quad \text{...(3.8)}$$

Equation (3.8) peaks when $\omega=K(\beta^2+\alpha^2)^{1/2}$ and the corresponding height of the spectrum is also obtained in the form

$$H(\omega,K) = K^2[2\alpha^2\beta^2(2K_\beta K_\alpha + K^2) - K^2(\alpha^4+\beta^4)$$
$$= K^2[4\alpha^2\beta^2 K_\beta K_\alpha - K^2(\alpha^2-\beta^2)] \quad \text{... (3.9)}$$

Thus, $H(\omega,K)$ favours waves of shorter length but essentially depends on the structural rigidity of the medium. Also, the peak speed might be likened to that of the body waves being greater than α and consequently β. However, in the case of the surface Rayleigh wave, the peak frequency has the representation of the form

$$\omega_m = \in K_R\left(\beta^2+\alpha^2\right)^{1/2}, \qquad 0.4 \le \in \le 0.6 \quad \text{..... (3.10)}$$

$\in$, thus, is an empirical factor obtained from the data of the different shallow Earth's structures.

If we employ Poisson's approximation, $\alpha^2 = 3\beta^2$, equation (3.9) simplifies to

$$H(\omega,K_R) = 2K_R^2\beta^2[6K_\beta K_\alpha - K_R^2] \quad \text{... (3.11)}$$

Equation (3.11) clearly depicts the dependence of $H(\omega,K_R)$ on β and essentially on rigidity.
But.

$$K_\beta = \frac{\omega}{\beta}, \quad K_\alpha = \frac{\omega}{\alpha} = \frac{\omega}{\beta\sqrt{3}}. \quad \text{... (3.12)}$$

Further application of Poisson's relation gives

$$H(\omega,K_R) = 2K_R^2[2\sqrt{3}\omega^2 - \beta^2K_R^2] \quad \text{.... (3.13)}$$

$H(\omega,K_R)=0$ when $\omega^2 = \dfrac{\beta^2K_R^2}{2\sqrt{3}}$ which is the critical frequency in the spectrum.

4. THE PHASE AND GROUP VELOCITY DECOMPOSITION

We now calculate the group velocity and in this case we shall use equation (3.4) in the form

$$\omega_2^2 = K^2\{(\alpha^2+\beta^2)+R_1\} \quad \text{.... (4.0)}$$

Differentiating with respect to K gives

$$\omega_2\frac{\partial\omega_2}{\partial K} = 2K[\alpha^2+\beta^2+R_1] + K^2\frac{\partial R_1}{\partial K} \quad \text{.... (4.1)}$$

$$\Rightarrow \quad \frac{\partial R_1}{\partial K} = \frac{8\alpha^2\beta^2K_\beta K_\alpha}{K^2} \quad \text{.... (4.2)}$$

Thus,

$$\frac{\partial R_1}{\partial K} = -\frac{4\alpha^2\beta^2 K_\beta K_\alpha}{K^3\sqrt{\alpha^2+\beta^2+R_1}}. \quad \text{.... (4.3)}$$

And,

$$\omega_1 \frac{\partial R_1}{\partial K} = \frac{K\{(\alpha^2+\beta^2)+R_1\}^{3/2} - 4\alpha^2\beta^2 K_\beta K_\alpha}{(\alpha^2+\beta^2+R_1)^{1/2}} \quad \text{.... (4.4)}$$

So, if V_1 is the group velocity and c the phase velocity, where $c = \frac{\omega}{K}$ and $V_1 = \frac{\partial \omega}{\partial K}$ we obtain the product

$$cV_1 = \frac{(\alpha^2+\beta^2+R_1)^{3/2} - 4\alpha^2\beta^2 K_\beta K_\alpha}{(\alpha^2+\beta^2+R_1)^{1/2}} \quad \text{...(4.5)}$$

Equation (4.5) simplifies further if we again apply Poisson's relation. Thus,

$$R_2 = \frac{12\beta^2}{K^2}\left(K^2 - K_\beta K_\alpha\right) \qquad R_1 = 2\beta^2\left[7 - \frac{K_\beta K_\alpha}{K^2}\right] \quad \text{... (4.6)}$$

$$\text{and} \quad cV_1 = \frac{(4\beta^2+R_1)^{3/2} - 12\beta^4 K_\beta K_\alpha}{(3\beta^2+R_1)^{1/2}}. \quad \text{...(4.7)}$$

Equation (4.7) suggests that $cV_1 = O(\beta^2)$ indicating the strong influence of the structural rigidity on the group velocity material dispersion. By appropriate change of sign, we obtain the identical expression for V_2 in the form

$$cV_2 = \frac{(4\beta^2+R_1)^{3/2} - 12\beta^4 K_\beta K_\alpha}{(3\beta^2+R_1)^{1/2}} \quad \text{... (4.8)}$$

Equation (4.7) and (4.8) suggests the form of the spectral decomposition of the group velocities. In this consideration, cV_1 is the lower frequency branch whilst cV_2 is the upper branch.

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