J. Nig Ass. Math. Phys.
Vol. 4 (2000)

STRESS ANALYSIS FOR AQ CRACKED CYLINDER UNDER LONGITUDINAL SHEAR LOADING

NNADI JAMES N
DEPARTMENT OF MATHEMATICS, ABIA STATE UNIVERSITY
UTURU, ABIA STATE

ABSTRACT

The problem solved anatically concerns the stress distribution in an elastic cylinder containing an edge crack and subjected to out-of-plane shear. Complete solution in a closed form is obtained for the displacement which in turn yields the expressions for the stresses everywhere in the material. The stress intensity factor and the energy release rate derived are proportional to that of a cracked semi-infinite elastic material subjected to the same mode of loading.

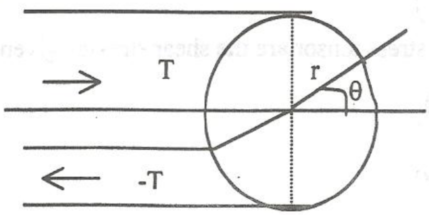

Fig 1: Geometry of the problem

1 INTRODUCTIÓN

Solutions of longitudinal shear problems for cracked infinite or semi-infinite bodies of various configurations are available (see for example Paris and Sih [1], Amazigo [2,3], Choi et al[4,]5) It is of interest to understand the stress distribution in a finite body configuration under longitudinal shear, since most practical situations are not modeled by infinite or semi-infinite body configuration. Several authors have addressed crack problems involving finite geometries for in-plane loadings and torsion by various analytic and numerical techniques (see for example Bowie [6] Bowie and Neal [7], Tweed and Rooke [8], Delale and Erodgan [9], Melrose and Tweed [10]). The analogous problem for longitudinal shear has received less attention; the only paper known to us in this direction is that of Westmann and Yang [11].

In this paper we consider an isotropic homogenous elastic circular cylinder occupying the region given in polar coordinates (r, θ, z) by

$$-\infty < z < \infty, \quad x = r\cos\theta, \quad y = r\sin\theta$$

$$r \leq a, \quad -\pi \leq \theta \leq \pi$$

with an edge crack of dept a along the segment

$\theta \pm \pi, \quad r \leq a$ (Fig. 1)

Equal and opposite uniformly distributed loads are applied parallel to the generators along the segments

$r = a,\ 0 \leq \theta \leq \pi$

and

$r \leq a,\ -\pi \leq \theta \leq 0$

respectively.

The crack surfaces are stress free.

2 MATHEMATICAL FORMULATION

The character of this mode of loading is that other components of the displacement vector vanish except the one in the z-direction which we denote by w. In terms of the polar coordinates, w satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial^1}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) w(r,\theta) = 0 \tag{1}$$

The non-zero components of the stress tensor are the shear stresses given by

$$\sigma_{\theta z}(r,\theta) = \frac{\mu}{r}\frac{\partial w}{\partial \theta}(r,\theta) \tag{2}$$

$$\sigma_{rz}(r,\theta) = \mu\frac{\partial w}{\partial \theta} = (r,\theta) \tag{3}$$

where μ is the shear modulus.

The following boundary conditions arise

$$\sigma_{rz}(a,\theta) = T, \qquad 0 < \theta < \pi \tag{4}$$

$$\sigma_{rz}(a,\theta) = -T, \qquad -\pi < \theta < 0 \tag{5}$$

$$\sigma_{\theta z}(r,\pm\pi) = 0 \qquad r \leq a \tag{6}$$

3 A TRANSFORM PLANE PROBLEM

We use the conformal mapping function

$$\zeta(z) = \frac{4\frac{z}{a}}{\left(1-\frac{z}{a}\right)^2}, \qquad z = x + iy \tag{7}$$

to transform the problem into a ζ-plane with a slit along the segment $-\infty < \mathrm{Re}\zeta \leq 0$.

The appropriate inverse is

$$\frac{z}{a} = 1 - \frac{2}{1 + (1+\zeta)^{1/2}} \tag{8}$$

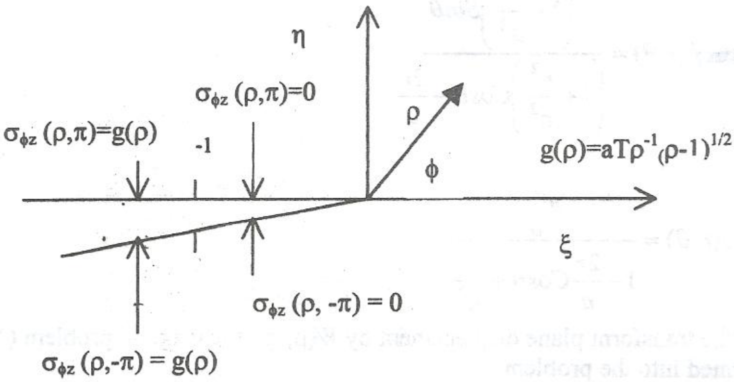

FIG II

Let

$$\zeta(r,\theta)=\xi(r,\theta)+i\eta(r,\theta)$$

then

$$\zeta(r,\theta)=\frac{4\frac{r}{a}\left[\left(1-\frac{r^2}{a^2}\right)Cos\theta-2\frac{r}{a}\right]}{\left[1+\frac{2^{r}}{2\frac{r}{a}}Cos\theta+\frac{r^2}{a^2}\right]^2}$$

$$\eta(r,\theta)=\frac{4\frac{r}{a}\left[\left(1+\frac{r^2}{a^2}\right)Sin\theta\right]}{\left[1-\frac{2r}{a}Cos\theta+\frac{r^2}{a^2}\right]^2}$$

Introducing polar coordinates (ρ, ϕ) in the ζ-plane through

$\xi=\rho\ Cos\theta,\qquad \eta=\rho\ Sin\theta$

then

$$\tan\phi(r,\theta) = \frac{\left(1-\frac{r^2}{a^2}\right)Sin\theta}{\left(1+\frac{r^2}{a^2}\right)Cos\theta - \frac{2r}{a}} \qquad (9)$$

and

$$\rho(r,\theta) = \frac{4\frac{r}{a}}{1-\frac{2r}{a}Cos\theta + \frac{r^2}{a^2}} \qquad (10)$$

Denoting the transform plane displacement by $W(\rho, \phi)$, the original problem (1), (4) – (6) is transformed into the problem

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial^1}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)w(\rho,\phi) = 0 \quad \rho \geq 0,\ -\pi \leq \phi \leq \pi \qquad (11)$$

$$\sigma_{\phi z}(\rho,\pm\pi) = \begin{cases} 0 & ,\rho < 1 \\ aT\rho^{-1}(\rho-1)^{-\frac{1}{2}} & ,\rho > 1 \end{cases} \qquad (12)$$

where z is the longitudinal axis through the origin of the transform plane (Fig II).

The asymptotic behaviour of the stresses as $\rho \to 0$ is obtained by assuming a product solution of the form

$$W(\rho,\phi) = \Phi(\phi)\rho^k \qquad (13)$$

substituting (13) into (11) yields the ordinary differential equation

$$\left(\frac{d^2}{d\phi^2} + k^2\right)\Phi = 0$$

The relation

$$\sigma_{\phi z}(\rho,\phi) = \frac{\eta}{\rho}\frac{\partial w}{\partial\phi}(\rho,\phi) \qquad (14)$$

together with (12a) and the solution of the differential equation lead to the behaviour

$$W(\rho,\phi) = A_0\rho^{\frac{1}{2}}Sin\frac{\phi}{2} \qquad \text{as} \qquad \rho \to 0$$

Hence

$$\sigma_{\phi z}(\rho,\phi) = \frac{\eta}{2}A_0\rho^{\frac{1}{2}}Cos\frac{\phi}{2} \qquad \text{as} \qquad \rho \to 0$$

A Laurent series expression valid for $\rho > 1$ from (12b) is given by

$$\sigma_{\phi z}(\rho,\pm\pi) = aT\rho^{-1}\left(\rho^{-\frac{1}{2}} + \frac{1}{2}\rho^{-\frac{3}{2}} + \frac{1\cdot3}{2\cdot4}\rho^{-\frac{5}{2}} + \ldots\right)$$

Therefore

$$\sigma_{\phi z}(\rho, \pm\pi) = aT\rho^{-3/2} \qquad \text{as } \rho \to 0 \tag{15}$$

To obtain the general case for the coordinate ϕ as $\rho \to \infty$ we assume a solution of the form

$W(\rho, \phi) = O(\rho^{k})$ as $\rho \to \infty$

Then by (14)

$\sigma_{\phi z}(\rho, \phi) = O(\rho^{k-1})$ as $\rho \to \infty$

That is

$\sigma_{\phi z}(\rho, \pm\pi) = O(\rho^{k-1})$

which compared to (15) yields $k - 1 = -\frac{3}{2}$ or $k = -\frac{1}{2}$

Hence

$W(\rho, \phi) = O(\rho^{-1/2})$ as $\rho \to \infty$

4 SOLUTION OF THE TRANSFORM PLANE PROBLEM

The Mellin transform of $W(\rho, \phi)$ is defined by

$$M(W(\rho,\phi);s) = \overline{W}(s,\phi) = \int_0^\infty W(\rho,\phi)\rho^{s-1}d\rho, \quad -\tfrac{1}{2} < \operatorname{Re} s < \tfrac{1}{2} \tag{16}$$

with which (11) and (12) are transformed into

$$\left(\frac{d^2}{d\phi^2} + s^2\right)\overline{W}(s,\phi) = 0 \tag{17}$$

$$M(\rho\sigma_{\phi z}(\rho,\pm\pi);s) = \eta\frac{\partial \overline{W}}{\partial \phi}(s,\pm\pi) = aT\int_1^\infty (\rho-1)^{-\frac{1}{2}}\rho^{s-1}d\rho \tag{18}$$

With the aid of formula 3.2513 of [12] we evaluate the integral in (18) as

$$\int_1^\infty (\rho-1)^{-\frac{1}{2}}\rho^{s-1}d\rho = B\left(\frac{1}{2}-s, \frac{1}{2}\right), \quad \operatorname{Re} s < \tfrac{1}{2}$$

where $B(\rho,q)$ is the beta function. A series solution obtained from formula 8.383 3 of [12] is

$$B\left(\frac{1}{2}-s, \frac{1}{2}\right) = \sum_{k=0}^{\infty}\frac{a_k}{k+\frac{1}{2}-s} \quad \operatorname{Re} s < \tfrac{1}{2}$$

where

$$ak = \begin{cases} 1 & , k = 0 \\ \dfrac{1}{k!2^k}\prod_{m=1}^{\infty}(2m-1) & , k > 0 \end{cases}$$

Therefore (18) yields

$$\frac{\partial \overline{W}}{\partial \phi}(s,\pm\pi) = \frac{aT}{\mu}\sum_{k=0}^{\infty}\frac{ak}{k+\frac{1}{2}-s} \quad \operatorname{Re} s < \tfrac{1}{2} \tag{19}$$

The solution of (17) is written as

$$W(s,\phi) = A(s)Sin\, s\phi + B(s)Cos\, s\phi \tag{20}$$

The displacement (W(ρ,φ) is then given by the inverse Mellin transform defined by

$$W(\rho,\phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{W}(s,\phi)\rho^{-s}\,d\rho,\ -\tfrac{1}{2} < c < \tfrac{1}{2}$$

which, in view of (20) becomes

$$W(\rho,\phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \{A(s)Sin\, s\phi + B(s)Cos\, s\phi\}\rho^{-s}\,ds,\ -\tfrac{1}{2} < c < \tfrac{1}{2}$$

Utilizing (20) we get

$$\frac{\partial \overline{W}}{\partial \phi}(s,\pi) = sA(s)Cos\,\pi s - sB(s)Sin\,\pi s$$

$$\frac{\partial \overline{W}}{\partial \phi}(s,-\pi) = sA(s)Cos\,\pi s + sB(s)Sin\,\pi s$$

Observing from (19) that

$\frac{\partial \overline{W}}{\partial \phi}(s,\pi) = \frac{\partial \overline{W}}{\partial \phi}(s,-\pi)$ we obtain B(s) = 0 and

$$A(s) = \frac{aT}{\mu s Cos\,\pi s}\sum_{k=0}^{\infty}\frac{a_k}{k+\frac{1}{2}-s}$$

Hence

$$W(\rho,\phi) = \frac{aT}{\mu 2\pi i}\int_{c-i\infty}^{c+i\infty} \rho^{-s}\frac{Sin\, s\phi}{sCos\,\pi s}\sum_{k=0}^{\infty}\frac{a_k}{k+\frac{1}{2}-s}ds,\ -\tfrac{1}{2} < c < \tfrac{1}{2} \tag{21}$$

To evaluate the integral in (21) by residue method, we note that the integrand has simple poles at $s = -(n-\frac{1}{2})$, n = 1, 2, 3, . . . and double poles at $s = (n-\frac{1}{2})$, n = 1, 2, 3. . .

The residues at $s = -(n-\frac{1}{2}),\ n = 1, 2, 3, \ldots$ *are*

$$\frac{2(-1)^{n+1}}{(2n-1)\pi}\rho^{n-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{a_k}{n+k}Sin(n-\tfrac{1}{2})\phi$$

while the residues at $s = (n-\frac{1}{2})$, n = 1, 2, 3, . . . are

$$\frac{2(-1)^{n+1}}{(2n-1)\pi}\left[\left\{a_{n-1}\left(\ln\rho + \frac{2}{2n+1}\right) - \sum_{\substack{k=0\\k\neq n-1}}^{\infty}\frac{a_k}{n-k+1}\right\}Sin(n-\tfrac{1}{2})\phi - \phi Cos(n-\tfrac{1}{2})\phi\right]\rho^{-n+\frac{1}{2}}$$

Jordan's lemma is then applied to (21) by closing the contour in the left half plane. Re*s* < 0 for ρ< 1 and closing the contour in the right half plane Re*s* > 0 for ρ> 1. The complete closed form solution of the problem is given by Cauchy's residue theorem as the convergent series:

$$W(\rho,\phi)=\frac{2aT}{\mu\pi}\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\sum_{k=0}^{\infty}\frac{a_k}{n+k}\rho^{n-\frac{1}{2}}Sin(n-\tfrac{1}{2})\phi, & \rho<1\\ \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\left[\left\{a_{n-1}\left(\ln\rho+\frac{2}{2n+1}\right)-\sum_{k=0}^{\infty}\frac{a_k}{n-k+1}\right\}Sin(n-\tfrac{1}{2})\phi - \phi Cos(n-\tfrac{1}{2})\right]\rho^{-n+\frac{1}{2}} & ,\rho>1\end{cases} \quad (22)$$

5 SOLUTION OF THE ORIGINAL PROBLEM AND CRACK TIP FIELDS

The solution for the displacement W(r , ϕ) = W(ρ,ϕ) is obtained from the general solution (22) by substitution of

$$\phi(r,\theta)=\tan^{-1}\left\{\frac{\left[\left(1-\frac{r^2}{a^2}\right)Sin\theta\right]}{\left[\left(1+\frac{r^2}{a^2}\right)Cos\theta-\frac{2r}{a}\right]}\right\}$$

and

$$\rho(r,\theta)=\frac{4\frac{r}{a}}{1-\frac{2r}{a}Cos\theta+\frac{r^2}{a^2}}$$

into (22) to obtain a closed form convergent series solution in terms of r and θ. The result is

$$W(\rho,\phi)=\frac{2aT}{\mu\pi}\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\sum_{k=0}^{\infty}\frac{a_k}{n+k}\rho^{n-\frac{1}{2}}Sin(n-\tfrac{1}{2})\phi, & \frac{2\text{r}}{\text{a}}(Cos\theta+2)-\frac{r^2}{a^2}<1,\ |\theta|\le\pi, 0\le r\le a\\ \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\left[\left\{a_{n-1}\left(\ln\rho+\frac{2}{2n-1}\right)-\sum_{\substack{k=0\\k\neq n-1}}^{\infty}\frac{(-1)^{n-1}}{2n-1}\right\}Sin(n-\tfrac{1}{2})\phi-\phi Cos(n-\tfrac{1}{2})\phi\right]\rho^{-n+\frac{1}{2}}, & \frac{2\text{r}}{\text{a}}(Cos\theta+2)-\frac{r^2}{a^2}>1\ |\theta|\le\pi,\ 0\le r\le a\end{cases} \quad (23)$$

Observe that the crack tip is in the interior of the closed curve whose polar equation is

$$\frac{2r}{a}(Cos\theta+2)-\frac{r^2}{a^2}=1 \quad |\theta|\le\pi, \quad r\le a$$

Therefore the crack tip fields are obtained by asymptotic analysis of (23a).

Now $\rho\to 0$ implies $r\to 0$ and $\rho^{1/2}\to 2\left(\frac{r}{a}\right)^{1/2}$ as $r\to 0$ the dominant term in(23a) yields the crack tip displacement as

$$W(r,\theta)=\frac{4aT}{\mu}\sum_{k=0}^{\infty}\frac{a_k}{1+k}\left(\frac{r}{a}\right)^{\frac{1}{2}}Sin\frac{\theta}{2} \quad \text{as } r\to 0$$

The conventional form of the displacement in terms of the stress intensity factor K_{III} is [1,13]

$$W(r,\theta)=\frac{2}{\mu}K_{III}\left(\frac{r}{2\pi}\right)^{\frac{1}{2}}Sin\frac{\theta}{2} \qquad (24)$$

where

$$K_{III}=\frac{4\sqrt{2}}{\pi}(\pi a)^{\frac{1}{2}}T \qquad (25)$$

with $\sum_{k=0}^{\infty}\frac{a_k}{1+k}=2$

Equation (25) indicates a correction factor, $\frac{4\sqrt{2}}{\pi}$, when compared to the stress intensity factor for a semi-infinite elastic material with an edge crack of depth a under our mode of loading (see for example [1,2,3]). The expressions for the stresses everywhere in the cylinder are readily obtained from (23) with the aid of (2)and (3). The crack tip shear stresses are obtained from the asymptotic result (24) and by use of (2) and (3).

The magnitude of the force per unit thickness P(T) acting on each loaded semi-circular segment is given by

$$P(T) = \pi aT$$

And is related to K_{III} through the expression

$$K_{III} = 4\sqrt{2}\pi^{-\frac{1}{2}}a^{-\frac{1}{2}}P(T)$$

The dependence of the ratio $\frac{K_{III}}{P(T)}$ on a is shown in fig III.

The energy release rate G is given by

$$G = \lim_{\Delta a\to 0}\frac{1}{\Delta a}\int_0^{\Delta a}\frac{1}{2}\sigma_{\theta z}(x,0)\{w(r,\pi)-w(r-\pi\}dx$$

where the crack advances by a small amount Δa and r is measured from the crack tip after the advance for which $0 < x < \Delta a$, $\theta = \pm\pi$ and $r = \Delta a - x$. we then obtain an integral which is evaluated to get

$$G = \frac{1}{2\mu} K_{III}^2$$
$$= \frac{32}{\pi^2} G_\infty$$

where G_∞ is the energy release rate for a semi-infinite elastic material subjected to longitudinal shear load.

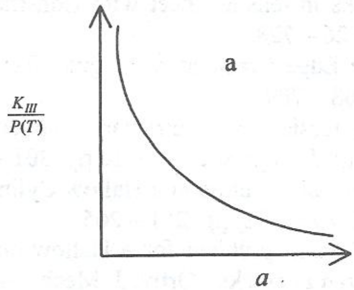

Fig III Dependence of $\frac{K_{III}}{P(T)}$ on a

REFERENCES

1. Paris P.C. and Sih G.C.; Stress Analysis of Cracks. Symposium on Fracture Toughness testing and its application. ASTM Special Technical Publication No. 381 (1965) pp. 30 – 83.
2. Amazigo J.C; Fully Plastic Crack in an Infinite Body under Anti-plane shear. Int J Solids structures Vol. 10 (1974) pp. 1003 – 1015.
3. Amazigo J.C; Fully Plastic Center – Cracked strip Under Antiplane Shear. Int. j. Solid structures Vol. 16 (1975) pp. 1291 – 1299.
4. Choi S.R. and Earmme Y.Y.; Analysis of a kinked Crack in Antiplane Shear. Mechanics of materials Vol. 9 (1990) pp. 529 – 533.
5. Choi SR, Lee KS and earmme YY; Analysis of a kinked Interfacial crack under ou-of-plane shear. J. Appl. Mech. Vol. 61 (1994) pp. 38 - 44.
6. Bowie o. Symmetric Edge Cracks in tensile Sheet with Constrained Ends. J. Appl. Mech. Vol. 31 (1964) pp. 726 – 728.
7. Bowie O. and DM Neal; Single Edge Crack in rectangular Tensile Sheet. J. Appl. Mech. Vol. 32 (1965, pp 708 – 709.
8. Tweed J and Pooke DP; The torsion of a circular cylinder containing a symmetric array of edge crack. Int. J. Engr Sec. Vol. 10 pp. 801 – 812.
9. Delale F and Edogen F; stress Intensity factors in a Hallow Cylinder containing a radial crack. Int. J Fracture vol. 20 (1982) pp. 251 – 265.
10. Melrose G and Tweed J; The Torsion problem for a Hallow circular cylinder containing a symmetric array of radial cracks. Qrtly. J. Mech, Appl. Maths Vol 42 (1989) pp. 289 - 301.
11. Westmann RA and Yang WH. Stress Analysis of Cracked Rectangular Beams. J. Appl. Mech. Vol. 34 (19670 pp. 693 – 701.
12. Gradshyteyn I.S.and Ryzhik I.M. Tables of integrals series and products. Academic Press, New York 1965.
13. Rice, J.R mathematical Analysis in the Mechanics of fracture in fracture. An advanced treatise (H. Liebowitz ed) vol II. Academic Press, New York, (1968) pp. 191 – 311.