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ON THE RELATIVE CONTROLLABILITY OF PERTURBATIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS WITH DISTRIBUTED DELAYS IN CONTROL

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ABSTRACT

The purpose of this study is to examine conditions for the relative controllability of perturbations of non-linear functional differential systems with distributed delays, including systems with implicit derivative in the perturbation function. The method of solution involves the linearization of the non-linear base. With enough smoothness conditions imposed on the perturbation function and an assumption of the non-singularity of the controllability map, the question of the relative controllability of the system under study, is settled in the affirmative using Darbo's fixed point theorem. The result extends Chukwu's efforts [10, 11] to systems with implicit derivative and carries over Balachandran's result in [6] to functional differential systems.

INTRODUCTION

Nonlinear systems present a challenging but fascinating area of study in mathematical control theory. They represent better approximations of real life dynamics and pose the obvious difficulty of not lending themselves readily to the standard, systematic and precise procedure of tackling controllability problems.

However, several studies have been conducted on perurbations of linear systems. In most of these studies results were obtained by placing boundedness and continuity conditions on the perturbation functions; and the Schauder's fixed point approach greatly in use. Dauer and Gahl [1], Onwuatu [2], Gahl [3], Chukwu [4] have independently shown that a linear perturbation is controllable provided, the linear base is controllable and the perturbation being bounded. Nonlinear systems have been studied by Klamka [9] and Onwuatu [5]. Balachandran and dauer [8], Dacka [7] and Balachandran [6] have considered perturbations of non-linear ordinary systems with implicit derivative. In these studies, together with the dynamics modelled by Chukwu [10, 11], the systems are multi-parameter dependent, necessitating the redefinition of the fundamental matrix solution and the controllability grammian to take care of these systems' varying arguments.

The use of Darbo's fixed point theorem in [6], [7], [8] imposed the calculation of the common modulus of continuity of functions in a set and consequently the measure of non-compactness of the set to take care of the rigours introduced by the presence of implicit derivative.

From the foregoing, the relative controllability of non-linear functional differential systems with distributed delays remains unsettled, especially for systems with implicit derivative, the investigation of which is the main objective of our research here.

1 NOTATION AND PRELIMINARIES

Let E = $(-\infty,\infty)$ and E^n be the n-dimensional Euclidean space with norm 1.1. The symbol $C = C([-h,o],E^n)$ denotes the space of continuous functions mapping the interval [-h,o], h > 0, h $\in E^n$ into E^n with the supremum norm $\|\bullet\|$ defined by

$$\|\phi\| = \sup|\phi(\theta)|; \phi \in C - h \le \theta \le 0$$

while $C^1 = C^1([-h,0],E^n)$ denotes the space of differentiable functions mapping the interval [-h,0] into E^n.

Let (X, $\|\bullet\|$) be a Banach space and O a bounded subset of X, the measure of non-compactness of Q,

$\mu(Q) = \inf\{r > 0 : Q$ can be covered by a finite number of balls of radii less than r}

For the space of continuous functions $C([t_0,t_1],E^n)$, the measure of non-compactness of a set Q is given by

$$\mu(Q) = \tfrac{1}{2} W_0(Q) = \tfrac{1}{2} \lim_{h\to 0^+} W(Q,h)$$

where W(Q,h) is the common modulus of continuity of the functions which belong to the set Q, that is

$$W(Q,h) = \sup_{x\in Q}\{\sup|x(t) - x(s)| : |t - s| \le h$$

For the space of differentiable functions $C^1([-h,0],E^n)$, we have

$$\mu(Q) = \tfrac{1}{2} W_0(DQ)$$

where

$$DQ = \{x : x \in Q\}$$

If t $\in [t_0,t_1]$ we let $x_t \in C^1$ be defined by $x_t(s) = x(t+s), s \in [-h,0]$

Also, for functions $u : [t_0 - h, t_1] \to E^m$, h > 0, and t $\in [t_0,t_1]$ then u_t denotes the functions on [-h,0] defined by $u_t(s) = u(t + s)$ for s $\in$ [-h,0].

The integral are the Lebesgue Stieltjes sense.

Consider the system of interest

$$\dot{x}(t) = L(t,x_t,u_t)x_t + B(t,x_t,u_t)u_t + f(t,x_t,\dot{x}(t)\, u(t)) \qquad (2.1)$$

With the following basic assumptions:

$$L(t,\psi)x_t = \int_{-h}^{0} d\eta(t,s,\phi,\psi)x(t+s)$$

where the $n \times n$ matrix function $\eta(t,s,\phi,\psi)$ is measurable in (t,s) $\in$ ExE, normalized so that

$\eta(t,s,\phi,\psi)=0; \quad s \geq 0$ for all ϕ,ψ

$\eta(t,s,\phi,\psi)=\eta(t,-h,\phi,\psi)$ for all $s \leq -h$

$\eta(t,s,\phi,\psi)$ is continuous from the left in s on [-h.,0] and has bounded variation in s on [-h,0] for each t, ϕ, ψ and there is an integrable function m such that

$| L(t,s,\phi,\psi)x_t | \leq m(t) \quad \|x_t\|$ for all $t \in (-\infty,\infty)$, $\quad \phi, x_t \in C^1 \quad \psi \in C$

We assume $L(t,s,\phi,\psi)$ is continuous. The n x m matrix $B(t,x_t,u_t)u_t$ given by

$$B(t,x_t,u_t)u_t = \int_{-h}^{0} d_s H(t,x(t+s),u(t+s))u(t+s)$$

is continuous on all the variables and is of bounded variation in s on [-h,0]. Also the function f is continuous and satisfies the Lipschitz condition in all its arguments. Enough smoothness conditions on L and f are imposed to ensure the existence of solution of system (2.1) and the conditions dependence of same on initial data.

Definition 2.1
The set y(t) = $\{x(t), x_t, u_t\}$ is said to be the complete state of system (2.1).

Definition 2.2
System (2.1) is said to be relatively controllable on $[t_0, t_1]$, if for every initial complete state y(t_0) and every $x_1 \in E^n$, there exists a control u(t) defined on $[t_0, t_1]$ such that the corresponding trajectory of system (2.1) satisfies $x(t_1) = x_1$.

Definition 2.3 (Darbo's fixed point theorem)
If S is a non-empty, bounded closed convex subset of X and P:S→S is a continuous mapping such that for any $Q \subset S$, we have

$\mu(pQ) \leq k\mu(Q)$

where k is a constant $0 < k \leq 1$ then P is a fixed point.

2 MAIN RESULTS

To solve the relative controllability problem for the system (2.1) we consider the linear approximation of

$$\dot{x}(t) = L(t,x_t,u_t)x_t + B(t,x_t,u_t)x_t \tag{3.1}$$

given by

$$\dot{x}(t) = L(t,z,v)x_t + B(t,z,v)u_t \tag{3.2}$$

where the arguments x_t, u_t of L and B have been replaced by specified functions $z \in C^1$ $v \in C$. System (2.1) can thus be approximated by

$$\dot{x}(t) = L(t,z,v)x_t + B(t,z,v)u_t + f(t,x_t,\dot{x}(t),u(t)) \tag{3.3}$$

For each $(z,v) \in C^1 \times C$. One can deduce the variation of parameter for system (3.3) using the unsystematic Fubini theorem.

Let X(t,s) = X(t,s,z,v) be a transition matrix for the system $\dot{x}(t) = L(t,z,v)x_t$ so that

$$\frac{\delta}{\delta}X(t,s) = L(t,z,v)X_t,s) \qquad (3.4)$$

where

$$X(t,s) = \begin{cases} 0 & s-h \le t < s \\ I & t = s \ (I \text{ identity}) \end{cases}$$

and where

$$X_t(\bullet,s)(\theta) = X(t+\theta,s) \quad -h \le \theta \le 0$$

The solution of system (3.3) is given by

$$x(t) = x(t,t_0,\theta,0) + \int_{t_0}^{t} X(t,s)\left[\int_{-h}^{0} d_\theta H(s,z(\theta),v(\theta)))u(s+\theta)\right]d_s \qquad (3.5)$$

$$+ \int_{t_0}^{t} X(t,s)f(s,x_s,\dot{x}(s),u(s))d_s$$

Using the unsymmetric Fubini theorem which gives impetus to the change of the order of integration, (3.5) can be written as

$$x(t_1) = x(t_1,t_0,\theta,0) + X(t_1,t_0) + \int_{t_0+s}^{t}\left\{\int_{-h}^{0} X(t_0,s-\theta)d_\theta\left[H(s-\theta,z(\theta),v(\theta)\right]t_{t_0}\right\}ds$$

$$+ X(t_1,t_0)\int_{t_0}^{t_1}\left\{\int_{-h}^{0} X(t_0,s-\theta)d_\theta\left[\overline{H}(s-\theta,z(\theta),v(\theta))\right]u(s)\right\}ds \qquad (3.6)$$

$$+ X(t_1,t_0)\int_{t_0}^{t_1} X(t_0,s)f(s,x_s,\dot{x}(s),u(s)ds$$

where

$$\overline{H}(s,z,v) = \begin{cases} H(s,z,v) & \text{for } s \le t \\ 0 & \text{for } s > t \end{cases}$$

Let us now define the following theorems at $t = t_1$

$$g(t_1) = g(Y(t_0),x(t_1),z,v) = x(t_1) - x(t_1,t_0,\phi,0)$$

$$-\int_{t_0}^{t_0}\left\{\int_{-h}^{0} X(t_1,s-\theta)d\theta\left[H(s-\theta,z(\theta),\dot{v}(\theta)\right]u_{t_0}\right\}ds \qquad (3.7)$$

$$-\int_{t_0}^{t_1} X(t_1,s)f(s,x_s,\dot{x}(s),u(s))ds$$

From

$$Z(t_0,s,z,v,0) = \int_{-h}^{0} X(t_0,s-\theta)d_0\overline{H}(s-\theta,z(\theta),v(\theta)) \qquad (3.8)$$

Thus, the controllability grammian of system (3.2) at time t_1 is

$$W(t_0,t_1) = W(t_0,t_1,z,v) = \int_{t_0}^{t_1} Z(t_0,s,z,v)Z^T(t_0,s,z,v)ds \tag{3.9}$$

where T denotes matrix transpose.

RELATIVE CONTROLLABILITY RESULTS

Given the system (3.3)

$$\dot{x}(t) = L(t,z,v)x_t + B(t,z,v)u_t + f(t,x_t,\dot{x}(t),u(t))$$

with conditions as spelt out above that L, B, f are continuous functions in all their variables; and that

$$|L(t, z, v,)x_t \leq m(t) \|x_t\| \tag{3.10}$$

where m(t) is an integrable function. B(t, z, v) is of bounded variation in s on [-h, 0]. The function f satisfies the Lipschitz condition with respect to the state variable, the response is uniquely determined by any control.

Furthermore

(a) $\| L(t, z, v) \| \leq M$ for each $s \in [-h,0]$

(b) $\| B(t, z, v) \| \leq N$ for each $s \in [-h,0]$

(c) $\| f(t, x_t, x(t), u(t) \| \leq K$ for each $t \in [t_0, t_1]$

$z \in C^1 \; u \in C([t_0,t_1],E^m)$

where M, N and K are some positive constants. Also, for every x, $x \in C^1 \; u \in C$ and $t \in [t_0,t_1]$

(d) $| f(t, x_t, x(t), u(t) - f(t, x_t, x(t), u(t) | \leq k \; | x(t) - x(t) |$ (3.11)

where k is a positive constant such that $0 < k \leq 1$

1 **Theorem 3.1**

Assume that inf det $W(t_0, t_1, z, v) > 0$ (3.12)

$z \in C^1$ then system (3.3) is relatively controllable on $[t_0, t_1]$.

Proof

Define the control u(t) for $t \in [t_0, t_1]$ as follows

$$u(t) = z^T(t_0,s,z,v)W^{-1}(t_0,t_1)g(Y(t_0),x(t),z,v) \tag{3.13}$$

where $Y(t_0)$ and $x(t_1) = x_1 \in E^n$ are chosen arbitrarily. The inverse of $W(t_0, t_1)$ is possible by condition (3.12). substituting (3.13) into (3.6) to replace u(t) and using (3.7) and (3.9) it is clear that the control u(t) defined by (3.13) steers the initial complete state Y(t) to the final state $x(t_1) = x_1 \in E^n$. The actual substituting of (3.13) into (3.6) yields

$$x(t_1) = x(t_1, t_0, \phi, 0) + \int_{t_0+s}^{t} \left\{ \int_{-h}^{0} X(t_0, s-\theta) d_\theta \left[H(s-\theta, z(\theta), v(\theta) \right] u_{t0} \right\} ds$$

$$+ \int_{t_0}^{t_1} \left(\int_{-h}^{0} X(t, s-\theta) d\theta \left[\overline{H}(s-\theta, z(\theta), v(\theta)) \right] x Z^T (t_0, s, z, v) W^{-1}(t_0, t_1) g(t_1) \qquad (3.14)$$

$$+ \int_{t_0}^{t_1} X(t,s) f(s, x_s, \dot{x}(s), u(s) ds$$

Consider the right hand side of (3.14) as nonlinear operator which maps the banach space $C^1\left([-h,0], E^n\right)$ into itself. Hence we can write (3.14) as

$$\dot{x}(t) = T(x)(t) \qquad (3.15)$$

This operator is continuous since all the functions involved in the operator are continuous.

Define the closed, convex subset G by

$$G = \{x : x \in C^1([-h,0], E^n, \|x\| \le N_1, \|Dx\| \le N_2 \qquad (3.16)$$

where the positive real constants N_1 and N_2 are given by

$N_1 = |\phi(t_0)| \exp M(t_1 - t_0) + a + (t_1 - t_0) b^2 c k_1 + k(t_1 - t_0) \exp M(t_1 - t_0)$

$N_2 = MN_1 + bcNk_1k_2 + k$

$k_1 = |x_1| + |\phi(t_0)| \exp M(t_1 - t_0) + a + k(t_1 - t_0) \exp 2m(t_1 - t_0)$

$k_2 = \max \text{ variation } H(t, s, z, v)$

$t\, s \in [-h, 0]$

$$a = \text{supremum} \left\| \int_{t_0}^{t} \left\{ \int_{-h}^{0} X(t_0, s-\theta) d_\theta \left[H(s-\theta, z(\theta), v(\theta) \right] u_{t0} \right\} ds \right\|; t \in [t_0, t_1]$$

$b = \sup \|z(t, s, z, v)\|; z \in C^1$

$c = \sup \|W^{-1}(t_0, t_1, z, v)\|$

The constants a, b, c and k_2 exist since the Lebesgue Stieltjes integral with respect to the variable θ is finite.

The operator T maps G onto itself. As clearly seen all the functions T(x(t) with x∈G are equicontinuous since they all have uniformly bounded derivatives. Now, we shall find an estimate of the modulus of continuity of the functions $DT(x)(t)$ for $t, s \in [t_0, t_1]$

$$|DT(x)(t) - DT(x)(s)| \le \left(m(t)\|x_t\| - m(s)\|x_s\| \right) + \int_{-h}^{0} d\theta H(t_1, z(t+\theta), v(t+\theta)) u(t+\theta) \qquad (3.17)$$

$$- d\theta H(s, z(s+\theta), v(s+\theta)) u(s+\theta) | + | f(t, x, \dot{x}, u) - f(s, x_s, \dot{x}, u0|$$

The first two terms of the right hand side of inequality (3.17) can be estimated as $\beta_0(|t-s|)$ where β_0 is a non-negative function s

$$\lim_{h\to 0} \beta_0(h) = 0$$

In the same manner, we find that the term of the right of (3.17) can be estimated from condition 3.11 (d) as

$$k|x(t)-x(s)|+\beta_1(|t-s|)$$

Letting $\beta = \beta_0 + \beta_1$, we finally obtain

$$|DT(x)(t)-DT(x)(s)| \leq k|x(t)-x(s)|+\beta(|t-s|)$$

Hence we conclude for any set Q∈G

$$\mu(TQ) \leq k\mu(Q)$$

Consequently, by the Darbo's fixed point theorem, the operator T has at least one fixed point, therefore, there exists a function $x^* \in C^1([-h,0], E^n)$ such that

$$x(t) = x^8(t) = T(x^*)(t) \qquad (3.18)$$

Differentiating with respect to t, we see that x(t) given by (3.18) is a solution to system (3.3); for the control u(t) given by (3.13). The control u(t) steers the system (3.3) from the initial complete state $Y(t_0)$ to the desired vector $x_1 \in E^n$ on the interval $[t_0, t_1]$ and since $Y(t_0)$ and x_1 have been chosen arbitrarily, then by definition 2.2 the system (3.3) is relatively controllable on $[t_0, t_1]$.

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REFERENCES

Dauer J.P and Gahl RD. Controllability by Non-linear Delay systems: Journal of **Optimization and applications** vol. 21, no. 1; (1977) pp. 59 – 69.

Onwuatu JU, Null controllability of nonlinear infinite Neutral system. KYBERNETIKA vol. 29, No. X, (1993).

Gahl RD. Controllability of nonlinear systems of neutral type: Journal of mathematical Analysis vol. 63 (1978) Pp. 33 – 42.

Chukwu EN. Total controllability to Affine Manifold of control systems. Journal of optimization theory and applications. Vol. 42, no. 2 (1984) Pp. 181 – 199.

Onwuatu JU On Controllability of nonlinear systems with distributed delays in the control. Indian Journal of Pure and Applied mathematics. Vol. 20, No. 3 (1984). Pp. 23 – 228.

Balachandran K Global relative controllability of nonlinear systems with time-varying multiple delay in control. International Journal of Control. Vol. 46, no. 1 (1987). Pp. 193 – 200.

Dacka C i.E.E.E. trans Auto. Control AC – 25 (1981).

Balachandran K and Daver JP relative controllability of perturbations of nonlinear systems Journal of Optimization theory and applications. Vol. 63, No. 1 (1989) Pp. 51 – 56.

Klamka J. Controllability of nonlinear systems with distributed delays in control. International Journal of control. Vol. 31 (1980) pp. 811 – 819.

Chukwu EN, Mathematical Controllability theory of the growth of wealth of nations. The japan Journal of Industrial and applied mathematics vol. 11, No. 1 91984), Pp. 87 – 111.

Chukwu EN. Control of global economic growth: Will the centre hold? (ed. J. Weiner and J.K. hale) Longman Scientific and technical (1992) Pp. 19 – 23.