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GENERALIZED TIME DEPENDENT ANALYTICAL SOLUTION OF BLOCH EQUATIONS FOR MAGNETIC RESONANCE IMAGING WITH TIME VARYING rF $B_1(t)$ MAGNETIC FIELD

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ABSTRACT

This study presents a generalized analytical solution of the Bloch equations which yield a time dependent signal that gives the fundamental relation of MR imaging. The final result of Fourier transformation of the total MRI signal is a set of images, each representing the spatial distribution of spins at a simple velocity (distance/time). In this way, the analysis provides techniques by which a magnetic resonance imaging sequence can be constructed so that the signal induced by acceleration and higher order terms of motion (which is a disturbing problem in the phase different method) can be eliminated or minimized.

Key words: Bloch equations, Magnetic Resonance, Pulse Sequence

INTRODUCTION

During the past two decades magnetic resonance (MR) has been proved to be a good non-invasive method of clinical diagnoses and prognoses. The information acquired with MR is of a physiological and morphological nature at molecular level, with respect to the metabolic processes in the human body.

Among its wide clinical applications, flow measurements are one of the highlights. For these reasons magnetic resonance has been rapidly applied in hemodynamics and fluid dynamics (1-6). The measurable flow-related information includes vessel geometries (angiography), and hemodynamical parameters, such as blood flow velocity, acceleration and jerk etc.

Flow measurements with magnetic resonance are based on three physical effects: Signal intensity modulation by inflow of fluids, artificial tagging (marking or labeling), flow induced phase shift. The first effect is mostly applied for the MR angiography (MRA). The tagging technique can produce impressive results for the direct visualization of the blood flow behaviour under slow and mediate flow conditions in relative simple vessel geometries, such as the cerebrospinal fluids (CSF) motion in the spine. The third effect, the phase shift (or phase contrast) method, is most promising (7-15) for the quantification of the blood flow pattern (velocity and acceleration patterns).

The phase shift technique describes motion in terms of Taylor series expansion. Often, the first three terms Taylor coefficients are used to describe the expanded motion. This is generally true for physiological flow in straight tubes or vessels. However, results of recent studies (16,17) show that even under physiological conditions, the sensitivity to higher order motion terms e.g. acceleration (m/s^2), pu7lsation (m/s^3) and other higher terms (m/s^n) of the Taylor expansion, can be very large and the combine effects due to these higher order terms of motion can reduce the signal strength of the acquisitions, thereby increasing the noise level in the phase difference image and falsify the velocity quantification (16-19). Though, there have been tremendous theoretical and practical advanced improvement of these MR flow techniques, new results of intensive current researches in this field are continuously being presented at scientific meetings. However a great deal of further research is needed to exhaust all the quantitative information for studying hemodynamics and fluid dynamics by magnetic resonance. An ideal approach to these further research would be to find generalized time dependent analytical solution to the Bloch NMR equations.

The advantages of the analytical solution are related to the fact that the magnetization and signal obtained are synthesis of many parameters that are of great clinical importance. Besides offering data about hydrogen density, it provides information through its T_1 and T_2 relaxation times about the chemical and physical environment in which the hydrogen protons reside.

The Bloch equations are coupled nonlinear equations describing the motion of a macroscopic magnetization 'M' under the influence of applied magnetic fields. Despite the volume of literature [20-26] regarding solutions to the Bloch equations, there are no simple closed solutions known for a general rF excitation. Therefore generalized time dependent analytical solution of the Bloch equation is obviously a very difficult task.

This study presents generalized time dependent analytical solution of Blooch equations for magnetic resonance imaging with time varying rF B_1 field. The analytical solution is then used to describe the nature of NMR-imaging signal detected. Next, we highlight one of the potential usefulness of the computations.

THEORY

For this investigation, we assume that resonance condition exists at Larmour frequency

$$f_0 = \gamma B - \omega = 0 \qquad (1)$$

In the following, γ is the gyromagnetic ratio of blood spins; $\omega/2\pi$ is the rF excitation of frequency, f_0/γ is the off-resonance field in the rotating frame of reference. They x,y,z components (in the rotating frame) of magnetization of a fluid bolus are given by the Bloch equation (20), which may be written as follows.

$$\frac{dM_x}{dt} = V \bullet gradM_x + \frac{\partial M_x}{\partial t} = \frac{-M_x}{T_2} \qquad (2)$$

$$\frac{dM_y}{dt} = V \bullet gradM_y + \frac{\partial M_y}{\partial t} = \gamma M_z B_1(t) - \frac{M_y}{T_2} \quad (3)$$

$$\frac{dM_z}{dt} = V \bullet gradM_z + \frac{\partial M_z}{\partial t} = -\gamma M_y B_1(t) + \frac{M_0 - M_z}{T_1} \quad (4)$$

Two reasonable' initial boundary conditions that may conform to the real-time experimental arrangements are chosen. These are

1. $M_0 \neq M_z$

a situation which holds good in general and in particular when the rF $B_1(x)$ **field is strong** say of the order of 1.0G or more so that M_z of the fluid bolus changes **appreciably form** M_0.

2. before entering the signal detection system, blood bolus has magnetization $M_x = 0$, $M_y = 0$,

From equations (3) and (4) we can write

$$V(x,t)^2 \frac{\partial^2 M_y}{\partial x^2} + 2V(x,t)\frac{\partial^2 M_y}{\partial x \partial t} + \frac{\partial^2 M_y}{\partial t^2} + \left(\frac{1}{T_1} + \frac{1}{T_2}\right)V(x,t)\frac{\partial M_y}{\partial x} + \left(\frac{1}{T_1} + \frac{1}{T_2}\right)\frac{\partial M_y}{\partial t}$$
$$+ \left(\gamma^2 B_1^2(t) + \frac{1}{T_1 T_2}\right) M_y = \frac{\gamma M_0 B_1(t)}{T_1} \quad (5)$$

Equation (5) is a general partial differential equation of order two in two independent variables, which can be represented as

$$Aa + 2Bb + Cc + Dd + Ee + Ff = Q \quad (6)$$

Where $A = V^2(x,t)$; $B = V(x,t)$; $C = 1$; $D = \left(\frac{1}{T_1 + T_2}\right)V(x,t)$; $E = \left(\frac{1}{T_1 + T_2}\right)$;

$$F = \left(\gamma^2 B_1^2(t) + \frac{1}{T_1 T_2}\right); \text{ and } Q = \frac{\gamma B_1(t) M_0}{T_1}.$$

A curve described by an equation of the form θ (x,t) = constant, where θ is a solution of the equation

$$A\left(\frac{\partial \theta}{\partial x}\right)^2 + 2B\frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial t} + C\left(\frac{\partial \theta}{\partial t}\right)^2 = 0 \quad (7)$$

is called a characteristics of equation (6).

Under the change of variable u = u(x,t), v = v(x,t) the partial derivatives of the first and of the second order with respect to x, and t are transformed in the following way:

$$e = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{yx} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = f_u u_t + f_v v_t \qquad (8a)$$

$$d = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_u u_x + f_v v_x \qquad (8b)$$

$$c = \frac{\partial e}{\partial t} = f_{uu}(u_t)^2 + 2f_{uv}u_t v_t + f_{vv}(v_t)^2 + f_u u_{tt} + f_v v_{tt} \qquad (8c)$$

$$b = \frac{\partial d}{\partial t} = f_{uu}u_x u_t + f_{uv}(u_x v_t + u_t v_x) + f_{vv}v_x v_t + f_u u_{xt} + f_v v_{xt} \qquad (8d)$$

$$a = \frac{\partial d}{\partial x} = f_{uu}(u_x)^2 + 2f_{uv}u_x v_x + f_{vv}(v_x)^2 + f_u u_{xx} + f_v v_{xx} \qquad (8e)$$

where u and v with the indices x and t denote the corresponding derivatives. Therefore, after the transformation of variables, equation (6) takes the form

$$A'f_{uu} + 2B'f_{uv} + C'f_{vv} + D'f_u + E'f_v + Ff = G \qquad (9)$$

where

$$A'(u,v) = A(u_x)^2 + 2Bu_x u_t + C(u_t)^2, \qquad (10a)$$

$$B'(u,v) = Au_x v_x + B(u_x v_t + u_t v_x) + Cu_t v_t \qquad (10b)$$

$$C' = A(v_x)^2 + 2B_x v_t + C(v_t)^2 \qquad (10c)$$

and x = x(u,v), t = t(u,v) is the transformation inverse to u = u(x,t). We do not write down the expressions for the other coefficients of equation (9) since they are of no interest for our present goals. Since equation (5) is parabolic, we take as the function u(x,t) a solution (7) different from a constant while the function v(x,t) should be chosen so that the condition

$$A(v_x)^2 + 2Bv_x v_t + C(v_t)^2 \neq 0$$

holds. By virtue of the equality $A(u_x)^2 + 2Bu_x u_t + C(u_t)^2 \neq 0$, we conclude from equation (10a) and (10b) that A' = B' = 0. Consequently, after all the terms of equation (9) have been divided by $A(v_x)^2 + 2Bv_x v_t + C(v_t)^2 \neq 0$, we obtain

Is a convolution integral and in general we may note that convolution with H(t) means integration. From equation (21), we can write

$$R(t) = \int_{-\infty}^{t} H'(t)dt'$$

$$R'(t) = H(t)$$

where $(\gamma B_1/n)(t)$ represents the velocity of n number of hydrogen protons per unit volume in the selected slice to which a steady pulse of $\gamma B_1 H(t)$ has been applied from the property that

$$H(t) * [f(t)H(t)] = \int_0^t f(t)dt$$

we can write

$$H(t) * [\gamma B_1 H(t)] = \gamma B_1 \int_0^t dt = \gamma B_0 \Delta t \qquad (22)$$

The pulse is based on the selection of rF amplitude B_1 and pulse width Δt to produce particular phase length such as 90^0 pulse which is

$$\gamma B_1 \Delta t = 90^0 \text{ and } \gamma B_1 \Delta t = 180^0$$

A 90^0 pulse will bend the magnetization from z direction to the x,y plane and a 180^0 pulse reverses its direction. If a single 180^0 pulse is applied to an NMR sample at resonance, it will invert the direction of magnetization so that instead of pointing in the applied field direction B_0, it points antiparallel to it. After achieving it's alignment opposite to the field, M_y will return to its orientation along the filed direction in accordance with the expression

$$M_y(t) = \lim_{\tau \to 0} \left[\left(1 - e^{-t/\tau}\right) H(t) \right] \qquad (23)$$

It is useful in this analysis to have continuous approximations to $M_y(t)$. The continuous approximation to $M_y(t)$ is plotted in Fig. 2. for $T_1 = 1.0$ s and $T_2 = 0.125$ s. when τ changes from 0.000005 s to 0.5 s. Thus, an approximations to $M_y(t)$ is

$$\widetilde{M}_y(t) = \left(1 - e^{-t/\tau}\right) H(t) \qquad (24)$$

In this approximation the signal size after each γB_1 Δt pulse builds up to the thermal equilibrium value H(t). However, there is no appreciable change in the magnetization for values of $\tau \approx \tau_0$. Since τ_0 is expected to vary for different biological materials, accurate understanding of magnetization or signals at τ_0 can be useful in designing a more accurate MRI seqtence for the estimation of Blood flow rates. In a liquid state system, typifying subjects of biological interest, motional-narrowing limits are reached and equation (20) applies to any homogeneous region. It is of interest to note from equation (20) that T_0 is exquisitely sensitive to both the mean-square strengths and the "correlation-time" dynamics of environmental interactions. Different tissues in a biological subject, and even different microscopic locale (e.g., intracellular of extracellular environment) in a given uniform tissue, will exhibit different T_0.

ARBITRARILY SHAPED INDEFINITELY BRIEF PULSE SEQUENCES

Since the response of an applied pulse may be scrutinized with an oscilloscope of the highest precision and time resolution, we must, of course, be prepared to keep the applied pulse duration shorter than the minimum set by the quality of the measuring instrument. The impulse symbol enables us to make abbreviated statements about arbitrarily shaped indefinitely brief pulses. An intimate relationship between impulse symbol and the unit step function follows from the property that

$$\int_{-\infty}^{t} \delta(t')dt' = 1$$

it t is positive but zero if t is negative. Hence

$$\int_{-\infty}^{t} \delta(t')dt' = H(t)$$

If we replace δ(t) by the pulse sequence $\tau^{-1}\,\mathrm{sinc}(t/\tau)$ we obtained a sequence that has curious property of not dying out to zero where $t \neq 0$; at any value of t not equal to zero the value oscillates without diminishing as $\tau \to 0$. The sequence serves perfectly well to define δ(t) for a reason that is in connection with the sifting property.

If we wish to excite the transverse slice at $\tau = 1$, the sinc wave function becomes a sharp rectangular frequency function after Fourier transformation, and vice versa. We need an rF signal containing frequency given by $f_1 < f < f_2$. So a frequency spectrum is needed which ideally has a constant amplitude for $f_1 < f < f_2$, and which has zero amplitude outside this region as shown in Fig. 3b. The Fourier transform of this spectrum in the time domain shows a harmonic signal at the gyro magnetic frequency, which is amplitude modulated with the sinc function. This signal must be presented to the transmitting coil to excite a slice. A problem is, however, that the signal has infinite duration, so in practical situation it has to be cut off. The greater the number of side lobes of the sinc function that are cut off, the more the frequency spectrum deviates from the ideal form of Fig. 3b and the more the slice also deviates. We will investigate and discuss the excitation in more details in our next studies. For now it is sufficient to realize that a slice has been excited (in our example of a transverse slice) and that the signals coming from the spins in this slice must be measured in such a way that their position is determined and an image can be generated.

CONCLUSION

A generalized time dependent analytical solution of the Bloch equations for magnetic resonance imaging with varying rF $B_1(t)$ magnetic field has been presented. The precessing magnetization, $M_y(t)$ is written explicitly time dependent to denote intrinsic T_0 decay. $M_y(t)$ is proportional to the physical density of precessing spins, but may be modulated by other excitation factors. The quantity measured are the spatial Fourier-transform components of precessing magnetization and the quantity presented for imaging by reconstruction is the spatial distribution of processing magnetization during data collection.

This theory however, introduces another relaxation parameter τ, which tends to zero as the signal rapidly increases as a function of time and then settle down to a steady value. Thus, suggests that $T_2 >> \tau$ in biological tissues. It may be interesting to investigate experimentally the variation of relaxation parameter τ and τ_0 in different materials especially biological tissues. Hopefully, the results presented in this work will encourage further studies to obtaining valuable physiological information about the human peripheral circulation thus far not accessible to other noninvasive methods.

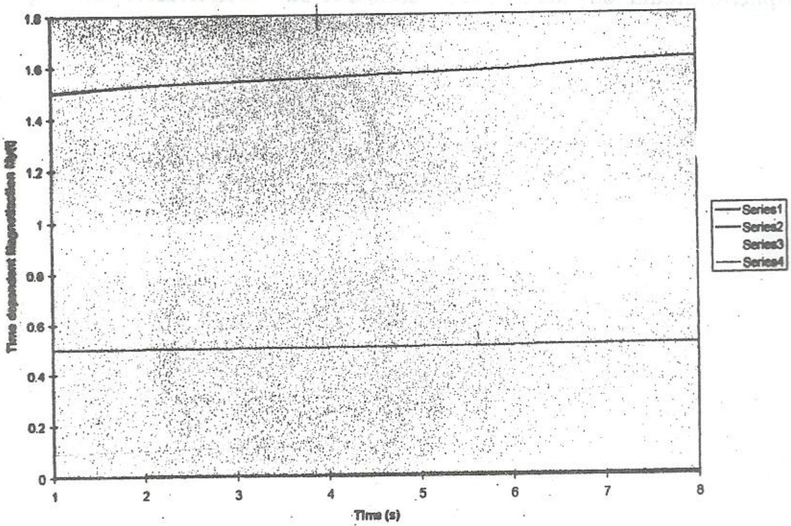

FIG. 1. Time dependent Magnetization $M_y(t)$ as a function of time according to equation (2.0) or (23). P = 4, T_1 = 1.0 s, T_2 = 0.125 s, 0.25 s, 0.5 s from top to bottom.

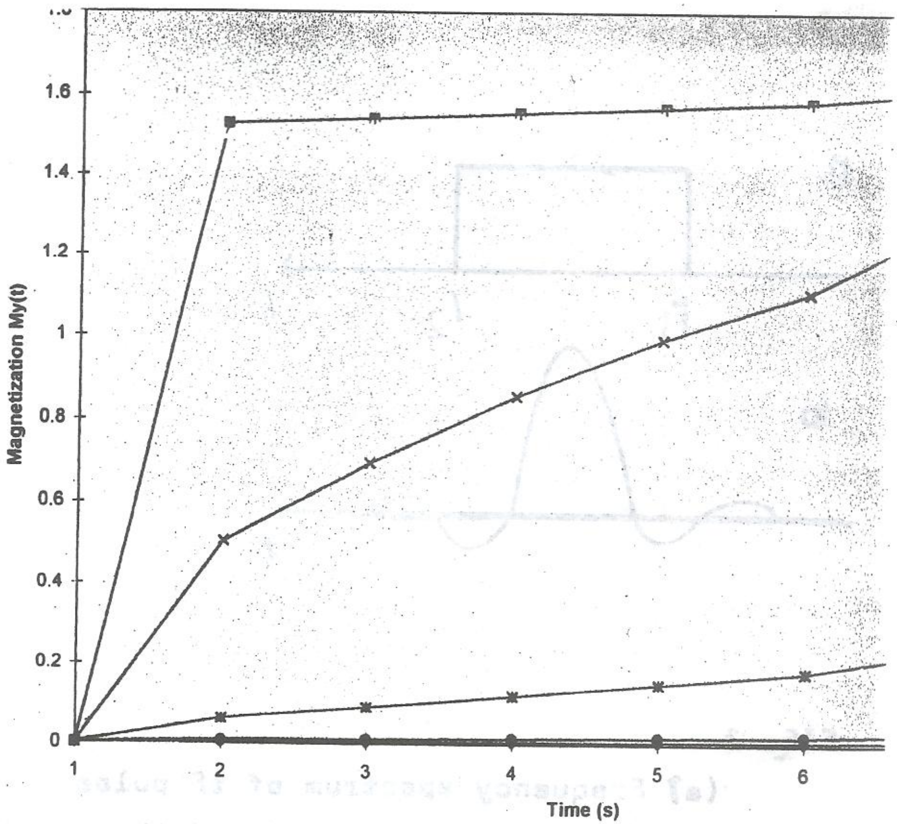

FIG. 2. Time dependent Magnetization $M_y(t)$ as a function of time according to equation (24). $P = 4$, $T_1 = 1.0$ s, $T_2 = 0.125$ s and $\tau = 0.000005$ s, 0.00005 s, 0.0005 s 0.005 s, 0.05 s, 0.5 s, from right to left

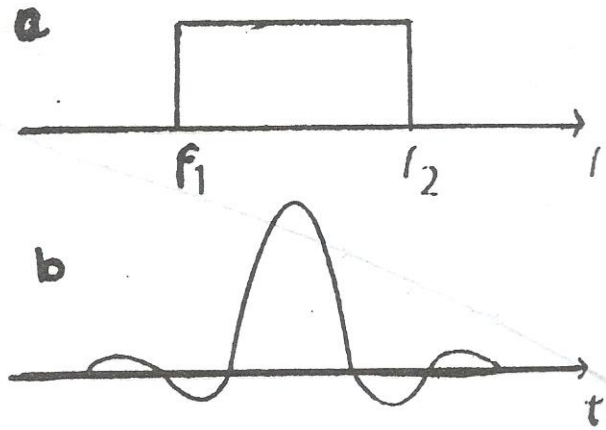

FIG. 3.

(a) Frequency spectrum of rF pulse

(b) Envelope of amplitude of the excitation pulse versus time

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