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ON A FAMILY OF METHODS FOR SIMULTANEOUS INCLUSIONS OF ZEROS OF POLYNOMIAL AND ANALYTIC FUNCTIONS INSIDE A SIMPLE SMOOTH CLOSED CONTOUR IN THE COMPLEX PLANE.

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ABSTRACT

A family of methods with k+2 R-Order which possesses high speed of convergence for the simultaneous determination of zeros of polynomial and analytic functions inside a simple smooth closed contour in the complex plane is considered. Neglecting terms involving analytic function in our method leads to the well-known method previously considered in [7]

Keywords: Simultaneous Methods, Analytic function, Bell's disk polynomial, interval arithmetic

1 INTRODUCTION

About twenty five years ago, Gargantini and Henrici came up in [3] with a very reliable iterative method for the simultaneous inclusion of a given polynomial in terms of circular region-disk.

This iterative method may be viewed as an interval extension of Ehrirch – Aberth third order method for polynomial in ordinary real floating point arithmetic.

Galius' famous theorem in the 16th century says that polynomial of degrees higher than four could not be solved by any known method involving the combination of coefficients of such polynomial. Since then many researchers have used numerical analysis techniques for polynomial whose degree $n \geq 3$. The problem of finding complex zeros of a polynomial has often been treated in the literature, but most results are of theoretical importance rather than practical reality. Applying some zero finding methods (often iterative in nature), it is essential to solve certain practical problem such as the following ones: computationally verifiable initial conditions providing a safe convergence of an applied algorithm, the construction of algorithm which possess a fast convergence in the presence of rounding errors, and information about error bounds of a complex approximation to the zero, etc.

The demands of the Computer age with it's finite precision have dictated the need for a structure which is referred to as interval arithmetic.

In particular error bounds procedures for solving certain problems in complex realm require complex interval arithmetic [5]. Three types of interval arithmetic exist namely, The complex, The Circular and The rectangular arithmetic [12].

The main advantage of these interval inclusion are:

i) The automatic determination of rigorous errors bounds for all approximate solutions at each iteration cycle,

ii) The ability to incorporate rounding errors without sacrificing the important inclusion property of the methods.

However, good we may portray interval arithmetic to be, its main disadvantage is the great computational complexity and the problem of wrapping effects caused by the over estimation of intervals. Although we may overcome the problem of wrapping effect in some cases.

In this paper we restrict our attention to the discussion of complex interval arithmetic for simultaneous determination for complex polynomials zeros

2 THE MAIN RESULTS: A GENERAL APPROACH

Let Z = Φ [z] be an analytic function inside and on the simple smooth closed contour Γ without zeros on Γ and with known numbers n of simple zeros inside Γ.

Then following approach of Smirnov [10], Φ will be of the form

$$\Phi(z) = X(z)\prod_{n}^{n}(z-\xi_j)^{\mu_j} \qquad (2.1)$$

Inside Γ, where $\xi_1\ \xi_2, \ldots, \xi_n$ are the zeros of Φ (inside Γ) and X is analytic function, but without Zeros inside Γ is determined by the argument principle (see Henrici [4] for more details)

$$n = \sum_{j}^{n} \mu j = 1/2\pi\,[\arg\Phi(t)]$$

$$= \frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi'(t)}{\Phi(t)}\,dt \qquad (2.2)$$

Where μj (j=1,2.......,n) are the number of multiplicities of Zeros such that number of zeros are exact. For simple zeros we have that

$\mu_1 = \mu_2 = \ldots\ldots = \mu_n = n$

Suppose that we have found disjoint complex disks $\xi_1, \xi_2, \ldots, \xi_n$, containing the zeros $Z_1, Z_2, \ldots\ldots, Z_n$ of Φ (Z) as

$$\xi_i = \{Z_i : |Z-C_i| \le r_j\} = \{C_j, r_i\} \qquad (2.3)$$

Where C = mid (Z).

The question that arises now is" 'how do we isolate these multiple zeros?" The suitable algorithm for locating such multiple zeros may be found in [2, 6, 7, 8,and 9].
The easiest and most portable is the Lagouall's limiting method enumerated in [4].

Denote ξ as an arbitrary point inside Γ such that $\Phi(\xi) \neq 0$. The analytic function X(Z) in line with Simirnov [10] is expressed in the form

$$X(Z)=\exp(F(Z)) \quad (2.4)$$

Inside Γ, where F is also analytic inside Γ, given by the relation [9] thus:

$$F(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\log(t-\xi)^{-n}\Phi(t)dt}{(t-z)} \quad (2.5)$$

$$F'(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\log(t-\xi)^{-n}\Phi(t)dt}{(t-z)^2} \quad (2.6)$$

$$F''(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\log(t-\xi)^{-n}\Phi(t)}{(t-z)^3}dt \quad (2.7)$$

The equations (2.6)and (2.7) can be integrated with the fact that $\xi=0$
Integration by parts applied to (2.6) and (2.7) respectively reveals that

$$F'(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi'(t)}{\Phi(t)}\frac{dt}{(t-z)} \quad (2.8)$$

$$F''(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi'(t)}{\Phi(t)}\frac{dt}{(t-z)^2} \quad (2.9)$$

The integrals (2.2), (2.6) and (2.7) may be computed with suitable accurate quadrature rule along the circumference. $\Gamma = \{z: |z| = R\}$ with nodes $z_{k_m} = R\exp(i\theta_{k_m})$,

$$\theta_{k_m}=\frac{(2k-1)\pi}{m}, \quad (k=1,2,\ldots\ldots,m) \quad (2.10)$$

In [4] trapezoidal rule was applied by the author along the circumference
To derive our method, let us first delineate our problem as follows:

$$= (z - \xi_i)\Delta_{k,i}(z) - \Delta_{k-1(z)} \qquad (2.15a)$$

If we set equation (2.15) equal to zero and rearranging common terms we obtain the so called Wang and Zhang method [11] with high speed of convergence k+1expressible as:

$$\xi_i = z_i - \left[\frac{\Delta_{k-1}(z_i)}{\Delta_{k+1}(z_i)}\right] \qquad (2.15b)$$

It is supposed that ξ_i (i=1,2,……,n) are reasonably close approximations to the requested zeros $z_1, z_2, \ldots, z_n$.

Therefore with regard to (2.14) it follows that

$$(z - \xi_i)\psi_{k,i}(z) = (z - \xi_i)\Delta_{k,1,i}(z) - \Delta_{K-1,I}(Z) \qquad (2.16)$$

Where from

$$\xi_i = z - \left[\frac{\Delta_{k-1,i}(z)}{\Delta_{k,i}(z) - \Delta_{k,i}(z)}\right] \qquad (2.17)$$

The generating function of $\Delta_{k,i}(z)$ is $\dfrac{P(z+x)}{P(z)}$ is that

$$\Delta_{k,i}(z) = \frac{(-1)^k}{k!}\left[D_x^k\left(\frac{P(z+x)}{P(z)}\right)^{-1/\mu}\right]_{X=0}$$

Using the rule of derivation of compound function we obtain the so called bell's polynomial [1].

$$\Delta_{k,i,}(z) = \sum_{v=1}^{n} ((-1)^{k-v} 1/\mu i\,(1/\mu + 1)\ldots(1/\mu + V - 1)$$

$$* \sum \prod_{\lambda=1}^{k}\left[(1/(q_\lambda !)P^{\lambda}(z)/\lambda!P(z)^{q\lambda}\right] \qquad (2.18)$$

Where the second sum on the right hand side runs over all non-negative integers $(q_1, q_2, \ldots q_k)$ which satisfy (2.11).
For simple zeros $\mu_1, \mu_2, \ldots, \mu_n = 1$ Thus $\Delta_{k,i}(z)$ takes the form recurrently

$$\Delta_{k,i,}(z) = \sum_{v=1}^{n} ((-1)^{v+1}/v!) * \left[P^{(v)}(z)/(P(z))\right]\Delta_{n-v,i}(z)$$

Besides, starting from

$$\Phi_{k,i}(z) = \frac{1}{k!}\Phi_i(z)^{1/\mu_i}\left[D_x^{(k)} \exp\left(\frac{-1}{\mu_i} \ln \Psi_i(z-x)\right)\right]_{x=0}$$

One obtains

$$\psi_{k,i}(z) = \sum_{v=1}^{k} \prod_{\lambda=1}^{k} (1/q\lambda!)\left(S_{\lambda,i}(z)\right)^{q\lambda} \tag{2.19}$$

Where

$$S_{\lambda,i}(z) = \frac{1}{\mu_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\mu_i/(z-\xi_i))^{\lambda}$$

And for the second sum the same is valid as in (2.18)
Besides, since

$$\psi_{k,i}(z) = \left[\left\{\left((-1)^{k-1}\right)/(\mu_i k!)\right\}\psi_i(z)^{1/\mu i}\right] * D_x^{(k-1)}\left[\psi(z)^{-1/\mu i}\psi'i(z)/\psi i(z)\right]$$

One obtains the following recursion relation

$$\psi_{k,i}(z) = 1/k \sum_{v=1}^{n} S_{v,i}(z)\psi_{k-v}(z)$$

It can be observed that the relation arising from Bells,s polynomial we have

$$\psi_{k,i}(z) = B_k(S_{1,i}(z), \ldots\ldots\ldots, S_{k,i}(z)) \tag{2.20}$$

If instead of using Bell's disk polynomial of (2.17), we may decide to consider method (2.1) in the following manner.

$$1/(z-\xi_i)=(1/\Phi(z))\exp(f(z))\prod_{\substack{j=1\\ j\neq 1}}^{n}(z-\xi_j)=H_i(z) \quad (2.21)$$

From the obvious identity here comes in handy mathematical logic thus

$$\xi_i = z+\frac{k\Delta_{k-1}\left(\frac{1}{z-\xi_i}\right)}{\Delta_k\left(\frac{1}{z-\xi_i}\right)} \quad (k=1,2,\ldots,) \quad (2.22)$$

Using the idea of equation (2.21) it follows that

$$\xi_i = z + k\,\Delta_{k-1}(H_i(z))/(\Delta_k(H_i(z))) \quad (2.23)$$

Logarithm differentiation of $H_i(z)$ in equation (2.21) leads to

$$d/dz \log Hi(z)=(H_i'(z))/(H_i(z))=F'(z)+\sum_{\substack{j=1\\ j\neq 1}}^{n}(z-\xi_j)-\frac{\Phi'(z)}{\Phi(z)} \quad (2.24)$$

For example we consider only two cases in this paper for k=1 and 2. Since any other values higher than k=2 are meaning less when compared in terms of computational complexities.
For k =1 we have

$$\xi_i = z-\frac{1}{\frac{\Phi'(z)}{\Phi(z)}-F'(z)-S_{1,i}} \quad (2.25)$$

Thus formula (2.25) provides a-construction of iterative method suitable for simultaneous determination of zero of analytic function inside a simple smooth closed contour in the complex plane. Method (2.25) is the well known Newton-like family method whose order of convergence is (k+2) Furthermore, for k = 2 we have the Halley like method:

$$\xi_i = z - \frac{2\left[S_{1,i}(z)+F'(z)-\frac{\Phi'(z)}{\Phi(z)}\right]}{\frac{\Phi'(z)}{\Phi(z)}-\left[\frac{\Phi'(z)}{\Phi(z)}\right]^2 - F''(z) + S_{2,i}(z) - \left[S_{1,i}(z)+F'(z)-\frac{\Phi'(z)}{\Phi(z)}\right]^2} \quad (2.26)$$

3. CONCLUSION AND DISCUSSION OF NUMERICAL RESULTS

From (2.25), it can be seen that the simplest case is the modified Newton method for k = 1 in (2.22) while for k = 2, we have the case of Haley like method. The order of convergence of these methods can be improved if we add the extra terms.

$N(z) = \Phi(z)/\Phi'(z)$ (Newton's Corrections)

$$H(Z) = \left[\frac{\Phi'(z)}{\Phi(z)} - \frac{\Phi''(z)}{2\Phi'(z)}\right]^{-1} \quad \text{(Halley Correction)}$$

into methods (2.25) and (2.26) that involve the terms $S_{k,l}(z)$

Again, the speed of convergence of these methods can be increased by using the principle of Total step method (TSM) and Single Step Method (SSM) known as Gauss – Siedel Method depending on the values of k.

In the light of Numerical evidence, the following polynomial equation is solved for k =1, while incorporating n(z), the Newton's correction.

$$Z_1^{(5)} - 6z^4 - 20z^3 + 120^2 + 64z - 384 = 0$$

with initial inclusion disks given as

$$Z_1^{(0)} = [-4.3, -3.9]\ Z_2^{(0)} = [-2.3, -1.9]$$
$$Z_2^{(0)} = [1.9, 2.3]\ Z_4^{(0)} = [3.9, 4.3]$$
$$Z_5^{(0)} = [5.9, 6.3]$$

The following results were obtained and are displayed in Table below for k =1

TABLE 1 : For k = 1

# of the iterations	Approximate result	
1	[-4.026841000,	-3.953256000]
	[-1.998403000,	-1.996853000]
	[1.989470000,	2.010819000]
	[3.989470000,	4.002468000]
	[5.940341000,	6.030243000]
2	[-4.000000000,	-4.000000000]
	[-2.000000000,	-2.000000000]
	[2.000000000,	2.000000000]
	[4.000000000,	4.000000000]
	[6.000000000,	6.000000000]

From Table 1, it can be seen that convergence was achieved just after two iterations. Other higher polynomial problems of order $n \geq 20$ have been tested with this method and are found to be accurate

Observe that the correction terms $S_{2,\ i}(z)$ and $S_{1,\ i}(z)$ involved in (2.26)enable us to construct simultaneous iterations and hence increase the method (2.26) from (k+1) to order (k+2). The construction of results obtained in Table 1 is based on the interval extension of the derivative of the polynomial Q'(z) inclusion monotonicity property of interval arithmetic is preserved.

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