A STEADY FLOW OF FLUID IN AN OPEN RECTANGULAR CONTAINER

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ABSTRACT

A steady slow flow of a viscous liquid in an open rectangular container is considered. The flow is driven by the base of the container which moves steadily along its plane. The top side of the container is left open with the liquid in contact with the air above it. Consequently the upper boundary of the flow is a free boundary and part of the unknown to be determined. Considering the inertial,gravitational and surface tension forces a numerical procedure for obtaining solutions for the cases when the capillary numbers are small is provided and the curves of the free boundaries obtained here presented for some flow parameters.

INTRODUCTION

The problem of a viscous liquid flowing in an open container is considered. (see fig 1).

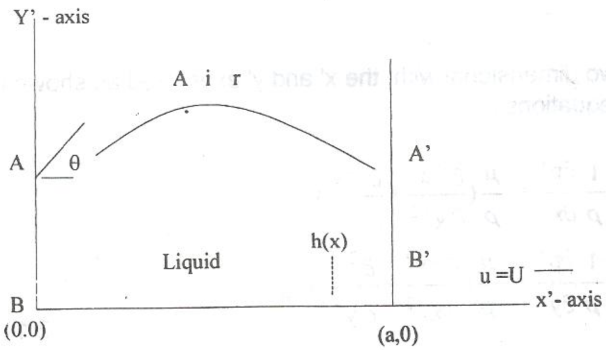

Fig. 1 b Liquid flow in an open container

The liquid is being driven by the container's base which moves along its plane. The upper part of the liquid is in contact with the air, (assumed to have zero shear stress) so that this part of the liquid is a free surface, the shape of which is part of the unknowns to be determined in the solution to the problem. The liquid is of

uniform density with the effect of gravity and surface tension considered. The contact angle of the surface at the walls are asusumed to be known.
The historical background to this problem can be found in [1,2,3] where the method of perturbation technique was used to obtain the first three leading approximations of the solution to the case when the flow is slow and thus governed by the Stoke's equation.

In this paper a report of the numerical experiments undertaken when not only the gravitation and the surface tension are taken into consideration but also when the inertial forces are included. The solution of the steady Navier stokes equations in two dimension must therefore be solved in this case.

In the numerical processes the physical domain, which is a varying one, is transformed into a fixed simple standard domain and it is in this domain that all the discretization takes place. The equations are turned into non-linear finite difference equations using implicit schemes and the resulting equations which are coupled are solved iteratively. With the normal stress linearized using Newton's method and use to revise the shape of the free boundary at each round of iterations the shapes of the free surfaces are obtained for some flow parameters.

By way of example the case where the container's base moves with a constant velocity U is considered.The singularities, this introduces at the corners B, B' was treated in [2],by obtaining an analytic solutions local to the regions of singularity and matching them to the numerical solution of the main flow (see [1,2] for similar treatment).

EQUATION OF FLOW

The equation of flow in two dimensional with the x' and y' axes fixed as shown in fig 1 are the Navier-Stokes equations :

$$u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}+\frac{1}{\rho}\frac{\partial p'}{\partial x'}=\frac{\mu}{\rho}\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}\right)$$

$$u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}+\frac{1}{\rho}\frac{\partial p'}{\partial y'}=\frac{\mu}{\rho}\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right)-g \qquad 2.1$$

where p', ρ, u' and v' are the pressure, density, horizontal and vertical components of the fluid's velocity respectively, g the acceleration due to gravity, and μ, the coefficient of viscosity, with

$$\nabla^2=\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}$$

assuming the flow is steady.
This must be supported by the equation of continuity,

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0 \qquad 2.2$$

The other equations are obtained from the no-slip and non- penetration conditions of the fluid at the solid walls

$$u'=v'=0,\text{at } x'=0,a \qquad 2.3a,b,c,d$$

$$v'=0,\ u'=Uf(x) \text{ at } y'=0 \qquad 2.4a,b$$

Kinematic and the stress conditions at the interface

y = h'(x) : (see [2,3,4]),

$$u'\frac{dh'}{dx'}-v'=0 \qquad 2.5$$

$$2\mu\left(\frac{\partial v'}{\partial y'}-\frac{\partial u'}{\partial x'}\right)_{h'}\frac{dh'}{dx'}+\mu\left[\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial x'}\right]_{h'}\left[1-\left(\frac{dh'}{dx'}\right)^2\right]=0 \qquad 2.6$$

$$2\mu\left[\left(\frac{dh'}{dx'}\right)^2\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}\right]-2\mu\left(\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial x'}\right)\frac{dh'}{dx'}=\frac{Y\dfrac{d^2h'}{dx'^2}}{\sqrt{1+\left(\dfrac{dh'}{dx'}\right)}} \qquad 2.7$$

the volume constraint:

$$\int_0^a h'dx'=\text{vol}' \qquad 2.8$$

and the interface contact angle constraint at the corners A and B

$$\left(\frac{dy'}{dx'}\right)_A=\tan\theta,\quad \left(\frac{dh'}{dx'}\right)_D=-\tan\theta \qquad 2.9a,b$$

The symbols and θ are the surface tension and the angle of contact of the interface with the wall.

DIMENSIONLESS VARIABLES

In terms of the dimensionless variables,

$$y = \frac{y'}{a},\quad x = \frac{x'}{a},\quad u = \frac{u'}{U},\quad p = \frac{ap''}{\mu U},\quad h = \frac{h'}{a'},\quad v = \frac{v'}{U}$$

with p" defined as

$$p'' = p' + gy'\rho$$

and

$$u = \frac{\partial\psi}{\partial y},\quad v = -\frac{\partial\psi}{\partial x}$$

the equations 2.1 to 2.9 become

$$\nabla^2\psi = \omega \qquad 3.1a$$

$$Re[\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}] = \nabla^2\omega$$

$$\psi = \frac{\partial\psi}{\partial x} = 0 \text{ at } x = 0,1 \qquad 3.2a,b,c,d$$

$$\psi = 0,\ \frac{\partial\psi}{\partial y} = 1 \text{ at } y = 0 \qquad 3.3a,b$$

$$-4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)_h \frac{dh}{dx} + \left[\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right]_h \left[1 - \left(\frac{dh}{dx}\right)^2\right]_h = 0 \qquad 3.4a$$

$$(-We\ p + Bh)\left[1 + \left(\frac{dh}{dx}\right)^2\right]_h + 2\frac{we}{Re}\left[\left(\frac{dh}{dx}\right)^2 - 1\right]\left(\frac{\partial^2\psi}{\partial x\partial y}\right)_h$$

$$+ 2\frac{We}{Re}\left(2\frac{\partial^2\psi}{\partial x^2} - \omega\right)_h \frac{dh}{dx} = \frac{d^2h}{dx^2}\left[1 + \left(\frac{dh}{dx}\right)^2\right]_h^{-\frac{1}{2}} \qquad 3.4b$$

$$\psi = 0 \text{ at } y = h \qquad 3.4c$$

$$\frac{dh}{dx} = \tan\theta \text{ at } x = 0 \qquad \frac{dh}{dx} = -\tan\theta \text{ at } x = 1 \qquad 3.5a,b$$

$$vol = \int_0^1 h\,dx \qquad 3.6$$

where

$$vol = \frac{vol'}{a^2}, Re = \frac{Ua\rho}{\mu}, We = \frac{aU^2\rho}{\gamma}, B = \frac{\rho g a^2}{\gamma}$$

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}$$

and p is to be obtained from the Navier-stokes equations.
The parameters vol, ca, B are the aspect ratio, capillary and Bond numbers respectively.They are dimensionless quantities.

NUMERICAL PROCEDURE

The physical domain [0,1]x [0,h(x)] is first transformed into the working domain [0,1]x[0,1] by the transformation

$$s = \frac{y}{h(x)}, \; t = x \qquad 4.1$$

In terms of these new variables s, t, the equations above become

$$\nabla^2 \psi = \omega \qquad 4.2a$$

$$Re[-\frac{1}{h}\frac{\partial \psi}{\partial t}\frac{\partial \omega}{\partial s} + \frac{1}{h}\frac{\partial \psi}{\partial s}\frac{\partial \omega}{\partial t}] = \nabla^2 \omega \qquad 4.2b$$

where

$$\nabla^2 = \frac{\partial^2}{\partial t^2} - 2\frac{s}{h}\frac{dh}{dt}\frac{\partial^2}{\partial t \partial s} + \frac{1}{h^2}[1 + (s\frac{dh}{dt})^2]\frac{\partial^2}{\partial s^2} + \frac{s}{h}[\frac{2}{h}(\frac{dh}{dt})^2 - \frac{d^2 h}{dt^2}]\frac{\partial}{\partial s}$$

and

$$\psi = \frac{\partial \psi}{\partial t} = 0 \text{ at } t = 0,1 \qquad 4.3$$

$$\psi = 0, \frac{\partial \psi}{\partial s} = h \text{ at } \quad s = 0 \qquad 4.4$$

$$\omega = -\frac{2}{h}\frac{d^2 h}{dt^2}\frac{\partial \psi}{\partial s}[1 + (\frac{dh}{dt})^2]^{-1}, \psi = 0 \text{ at } s = 1 \qquad 4.5a,b$$

$$We(P + p_0) - hB + \frac{We}{Re}\{2\frac{\partial}{\partial t}(\frac{1}{h}\frac{\partial \psi}{\partial s}) - \omega\frac{dh}{dt}\} = -\frac{\frac{d^2 h}{dt^2}}{\sqrt[3/2]{1 + (\frac{dh}{dt})^2}} \qquad 4.5c$$

$$\frac{dh}{dt} = \tan\theta \text{ at } t = 0, \frac{dh}{dt} = -\tan\theta \quad t = 1 \qquad 4.6a,b$$

The volume and the pressure equation become

$$\int_0^1 h \, dt = vol \qquad 4.6c$$

$$\frac{\partial p}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t}\{h^{-2}[1 + (\frac{dh}{dt})^2](\frac{\partial \psi}{\partial s})^2\} + \frac{1}{Re}\{\frac{1}{h}[1 + (\frac{dh}{dt})^2]\frac{\partial \omega}{\partial s} - \frac{dh}{dt}\frac{\partial \omega}{\partial t}\} \qquad 4.7$$

The working domain is divided into meshes (each of dimensions LxL) with the grid points at

$(t_i, s_j) \quad i = 1,........n, \quad j = 1,..........n$

where

$t_i = L_i, \quad s_j = (j + 0.5)L, \; L = 1/(n+1)$
and n is the number of meshes in a row.
Defining H, p as

$$H(x) = h(x) - h_0, \quad p = P - p_0$$

(where h_0 and p_0 are the height of the interface, and the non-dimensional pressure of the fluid, respectively at i= 0) and the derivatives in the equations above being replaced by their corresponding central difference operators, the following finite difference equations are obtained.

$$\psi_{i,j}^{k+1} = \psi_{i,j}^{k} + r[a_{i,j}\psi_{i+1,j}^{k+1} - 2(a_{i,j} + c_{i,j})\psi_{i,j}^{k+1} + a_{i,j}\psi_{i-1,j}^{k+1} + (c_{i,j} + d_{i,j})\psi_{i,j+1}^{k+1} + (c_{i,j} - d_{i,j})\psi_{i,j-1}^{k+1} + b_{i,j}(\psi_{i+1,j+1}^{k+1} - \psi_{i+1,j-1}^{k+1} - \psi_{i-1,j+1}^{k+1} + \psi_{i-1,j-1}^{k+1})] - L^2\omega_{i,j} \qquad 5.1a$$

$$\omega_{i,j}^{k+1} = \omega_{i,j}^{k} + r'[a_{i,j}\omega_{i+1,j}^{k+1} - 2(a_{i,j} + c_{i,j})\omega_{i,j}^{k+1} + a_{i,j}\omega_{i-1,j}^{k+1} + (c_{i,j} + d_{i,j})\omega_{i,j+1}^{k+1} + (c_{i,j} - d_{i,j})\omega_{i,j-1}^{k+1} + b_{i,j}(\omega_{i+1,j+1}^{k+1} - \omega_{i+1,j-1}^{k+1} - \omega_{i-1,j+1}^{k+1} + \omega_{i-1,j-1}^{k+1}) + f_{i,j}(\omega_{i+1,j}^{k+1} - \omega_{i-1,j}^{k+1})] \qquad 5.1b$$

$$a_{i,j} = 1, b_{i,j} = -\frac{1}{2}[\frac{s}{h_0 + H}\frac{dH}{dt}]_{i,j}$$

$$c_{i,j} = [\frac{1}{(H + h_0)^2}(1 + s^2(\frac{dH}{dt})^2)]_{i,j}$$

$$d_{i,j} = [\frac{Ls}{H + h_0}[(\frac{dH}{dt})^2\frac{1}{H + h_0} - \frac{1}{2}\frac{d^2H}{dt^2}] + \frac{1}{4}\frac{Re}{H + h0}\frac{\partial\psi}{\partial t}]_{i,j}$$

$$f = [-\frac{1}{2}\frac{Re}{H + h0}\frac{\partial\psi}{\partial s}]_{i,j}$$

The parameters r, r' are the relaxation parameters and $\psi_{i,j}^{k}$ is the approximate value of ψ at (i,j) on the k^{th} iteration (see [2] for detail).
The boundary conditions on the stream functions are approximated as

$$\psi_{0,j}^{k+1} = \psi_{n+1,j}^{k+1} = 0$$
$$\psi_{i,-1}^{k+1} = -\psi_{i,0}^{k+1}$$
$$\psi_{i,n+1}^{k+1} = -\psi_{i,n}^{k+1} \qquad 5.1c$$

while the conditions on the vorticity take the form

$$\omega_{0,j}^{k+1} = \left[\omega'_{0,j} - \omega_{0,j}^{k}\right]r + \omega_{0,j}^{k}$$
$$\omega_{n+1,j}^{k+1} = \left[\omega'_{n+1,j} - \omega_{n+1,j}^{k}\right]r + \omega_{n+1,j}^{k}$$
$$\tfrac{1}{2}[\omega_{i,0}^{k+1} + \omega_{i,-1}^{k+1}] = [\omega'_{i,1/2} - 0.5(\omega_{i,0}^{k} + \omega_{i,-1}^{k})]r + \tfrac{1}{2}[\omega_{i,0}^{k} + \omega_{i,-1}^{k}]$$
$$\tfrac{1}{2}[\omega_{i,n+1}^{k+1} + \omega_{i,n}^{k+1}] = [\omega'_{i,1/2+n} - \tfrac{1}{2}(\omega_{i,n+1}^{k} + \omega_{i,n}^{k})]r + \tfrac{1}{2}[\omega_{i,n+1}^{k} + \omega_{i,n}^{k}] \qquad 5.1d$$

where

$$\omega'_{i,1/2} = \frac{4}{9L^2(h_0 + H)^2}[27\psi_{i,0} - \psi_{i,1} - 12L(H_i + h_0)]$$

$$\omega_{i,n+\frac{1}{2}} = \frac{4[H_{i+1} - 2H_i + H_{i-1}]}{L^3[H + h_0]\left[1 + \left(\frac{H_{i+1} - H_{i-1}}{2L}\right)^2\right]}\psi_{i,n}$$

$$\omega'_{0,j} = \omega_{0,j} = \frac{1}{2L^2}\left[8\psi_{1,j} - \psi_{2,j}\right] - \frac{s_j \tan\theta\left[4\psi_{1,j+1} - 4\psi_{1,j-1} - \psi_{2,j+1} + \psi_{2,j-1}\right]}{2hL^2}$$

$$\omega_{n+1,j} = \omega'_{n+1,j} = \frac{[8\psi_{n,j} - \psi_{n-1,j}]}{2L^2} - \frac{s_j \tan\theta[4\psi_{n,j+1} - 4\psi_{n,j-1} - \psi_{n-1,j+1} + \psi_{n-1,j-1}]}{2hL^2}$$

and r is a relaxation parameter.

The corresponding finite difference equations for the pressure and the Normal stress equations are obtained, by first integrating eqs. 4.7 and 4.5c with respect to t, and approximating, their integrals that cannot be carried out analytically by trapezoidal integration formulars. If thereafter the derivatives are replaced by their central difference approximations then the following equations result.

$$p_i = L\left\{\sum_{j=1}^{i-1}[(1+(\frac{H_{j+1} - H_{j-1}}{2L})^2)\frac{(\omega_{j,n+1} - \omega_{j,n})}{(H_j + h_0)L} + \frac{(\omega_{j,n+1} - \omega_{j,n})}{2}\frac{(H_{j+1} - 2H_j + H_{j-1})}{L^2}\right.$$
$$+ \frac{L}{2}\left[1 + (\frac{H_{i+1} - H_{i-1}}{2L})^2\frac{(\omega_{j,n+1} - \omega_{j,n})}{(H_i + h_0)L} + \frac{L}{2}[1 + \left(\frac{(H_{i+1} + H_{i-1})}{2L}\right)^2\frac{(\omega_{i,n+1} - \omega_{i,n})}{(H_i + h_0)L}\right]$$

$$\frac{1}{2}\frac{(\omega_{i,n+1}-\omega_{1,n})(H_2-2H_1+H_0)}{L^2}$$

$$-\left[\frac{(\omega_{i,n+1}-\omega_{i,n})}{4}\left(\frac{H_{i+1}-H_{i-1}}{L}\right)^2+\left(\frac{\omega_{1,n+1}+\omega_{1,n}}{4}\frac{(H_2-H_0)}{L}\right]\frac{1}{Re}$$

$$+\frac{1}{2}[1+\left(\frac{(H_2-H_0)}{21}\right)^2[\psi_{1,n+1}-\psi_{i,n}]^2[H_i+h_0]^{-2}L^{-2}$$

$$-\frac{L}{2}[1+\left(\frac{H_{i+1}-H_{i-1}}{21}\right)^2]\left[\psi_{i,n+1}-\psi_{i,n}\right)^2\left[H_i+h_0\right]^{-2}L^{-2}\ \Big\}\qquad 5.2$$

$$L(P_i+P_0)We-L(H+h_0)B+\left[\frac{\psi_{i+1,n+1}-\psi_{i+1,n})}{(H_{i+1}+h_0)L}-\frac{\psi_{i-1,n+1}-\psi_{i-1,n}}{(H_{i-1}+h_0)L}\right]\frac{We}{Re}-$$

$$\frac{We}{Re}\left(\frac{\omega_{i,n+1}+\omega_{i,n}}{4}\right)(H_{i+1}-H_{i-1})+\frac{H_{i+1}-H_i}{L\sqrt{1+\left(\frac{H_{i+1}-H_i}{L}\right)^2}}-\frac{(H_i+H_{i-1})}{L\sqrt{1+\left(\frac{H_i+H_{i-1}}{L}\right)^2}}=0$$

5.3a

The boundary conditions on h (after replacing the derivatives involved in 4.6a by their corresponding third order finite difference approximations) become

$$18H_1-9H_2+2H_3=6L\tan\theta \qquad 5.3b$$

$$-11H_{n+1}+18H_n-9H_{n-1}+2H_{n-2}=6L\tan\theta \qquad 5.5c$$

$$\frac{1}{2}H_{n+1}+\sum_{i=1}^{n}H_i=(vol-h_0)/L$$

5.3d

The last equation (i.e eq. 5.3d) is obtained from 4.6c, on the application of the trapezoidal rule on its left hand side expression.

Finally the above finite difference equations are solved iteratively in the following manner.

(i) Initiate ω and ψ with zero and set h equal some reasonable value (say vol).

(ii) Perform iteration on (5.1a) until convergence incorporating the boundary conditions in 5.1c.

(iii) Compute the boundary condition on ω from (5.1d).

(iv) Perform iterations on 5.1b until convergence, incorporating the boundary conditions obtained from (iii).

(v) Check for the global convergence (for termination) and terminate iteration if necessary, otherwise proceed to the next step.

(vi) Solve for the pressure at the free surface from 5.2.
(vii) Solve for h and p_0 from eq. 5.3a to 5.3d and go back to step (ii).

The global convergence criteria (mentioned above) used as termination condition is

$$|H_i^{k+1} - H_i^k| < \varepsilon$$
$$|\psi_{i,j}^{k+1} - \psi_{i,j}^k| < \varepsilon$$
$$|\omega_{i,j}^{k+1} - \omega_{i,j}^k| < \varepsilon$$

where ε is taken as 0.0005.

RESULTS AND OBSERVATIONS

The above algorithm was implemented and the results based on meshes length 1/22 are recorded here. Tests on square meshes of lengths l=1/11,1/22,1/44 with extrapolation techniques and double precision arithmetic used were also undertaken to check for the consistency,the effect of round-off and truncation errors on the result so obtained [2].
The linear system resulting from the Normal stress equation (linearized using Newton's method) was solved at each round of iteration by Gauss-elimination method. The choice of the normal stress equation for revising the free surface is made due to the fact that we are interested in the flow with low capillary number (that is, a flow with large surface tension).
The algorithm however fails to converge when vol is small (for example for vol=0.2).
Although the flow depends on five parameters namely the We, Re,B,Q, and vol as defined above, the deviation of the shape of the interface from that of the corresponding hydrostatic problem is observed to be strongly dependent on the aspect ratio. The larger is the aspect ratio, the smaller is this deviation.
The dependence of the deviation of the interface is so strong that the interface suffers a sign reversal of its curvature for large values of Vol.

CONCLUSION

The work above considers a viscous flow with a free surface in a rectangular trough and provides an algorithm for obtaining its solution for the case when the inertial forces are considered.

The algorithm diverges for very low values of vol for any reasonable values of We and Re. The range of values for We/Re, for which the algorithm converges appears to be related to Vol. The larger is the aspect ratio, the larger is this range of values of ca for which convergence is possible.

The flow depends on five parameter, however the deviation of the interface curve from that of the corresponding hydrostatic problem, is strongly dependent on the aspect ratio, Vol.

APPENDIX A

Curvature Y_0, at a corner

B	Y_0
0.5	0.5116890
1.0	0.37610315
2.0	0.28565947
4.0	0.2286025
10	0.1897671
20	0.1785096
64	0.1746483

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