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NUMERICAL SOLUTION FOR A NON- BOUSSINESQU RADIATIVE FLOW IN THE ANNULUS OF CONCENTRIC ROTATING SPHERES

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ABSTRACT

The thermal properties like the viscosity, conductivity and difussibility of certain fluid have been shown by experiments to vary with its thermal state like its temperature and pressure. For instance in Birds et al [1] the viscosity of gasses at low density has been found to vary almost directly with the square root of its absolute temperature whereas the viscosity of a pure liquid was shown to decrease exponentially with its absolute temperature. In most of the familiar researches carried out on fluid flow the analysis are based on the invariance of this properties. The results obtained therefore from such analyses are bound to be unreliable in view of the variance of the principles on which such analyses are based with real fluid . This paper therefore carry out a numerical investigation of the flow generated by a fluid whose viscosity, conductivity and difussibility vary with its absolute temperature according to a model in the annulus of a rotating sphere. The results of this analysis is compared with the results obtained from Newtonian analysis(constant properties).

INTRODUCTION

At present there is considerable amount of work on the analysis of fluid flow in a sphere . Of particular relevance are those of Bestman [2] whose research was on the flow of gas in the annulus of a concentric sphere and that of Viskanta [3] who carried out an investigation on the flow of a radiating fluid in a duct.
Unfortunately in the literature sited above and most others in this field of study the problem of the variability of the fluid properties is not addressed. Most problems of practical application however point to the variability of these properties .The present paper therefore considers the effect of this phenomenon on the flow field. Hence the viscosity μ and the thermal conductivity κ are modeled as linear functions of the temperature of the fluid. The Cogley [4] approximation is assumed.

The mathematical model therefore consists of a set of coupled quasi linear partial differential equation describing the momentum balance and non - linear energy equation .The asymptotic approximation for small Reynold number is adopted. The solution of the resulting differential equations is effected via the Bobnov- Gerlakin procedure as developed in Ames[5]which leads ultimately to non - linear algebraic equation in the energy equation and linear algebraic system in the velocity equations for the order (1) problem. These are finally solved through the Newton - Ralphson iterative procedure developed for system of non – linear algebraic equation in Atkinson[6]and the Gaussian elimination technique respectively.

The analysis shows that for real system (variable properties) the results of the energy are higher and that of the velocity lower than when the properties are assumed constant .

2 MATHEMATICAL FORMULATION

At the time t = 0 the concentric spheres of inner radius r_0 and an outer radius of r_1 are set at constant angular velocities ω_0 and ω_1 and maintained at constant temperatures T_0 and T_1 respectively . The components of the fluid velocities being given as V_r ,V_θ and V_ϕ. In particular $r_1 = 2\ r_0$

Under the proposed model the governing differential equations are:

$$\frac{1}{r'^2}\frac{\partial(r'V_{r'})}{\partial r'}+\frac{1}{r'\sin\theta}\frac{\partial(V_\theta\sin\theta)}{\partial\theta}=0 \quad \ldots(2.1)$$

$$\rho\left[\frac{\partial V_{r'}}{\partial t}+V_r'\frac{\partial V_{r'}}{\partial r'}+\frac{V_\theta}{r'}\frac{\partial V_{r'}}{\partial\theta}-\frac{V_{r'}^2+V_\theta^2}{r'}\right]$$
$$=-\frac{\partial P}{\partial r'}-\frac{1}{r'^2}\frac{\partial\left(r'^2\tau_{r'r'}\right)}{\partial r'}+\frac{1}{r'\sin\theta}\frac{\partial(\tau_{r\theta}\sin\theta)}{\partial\theta}+\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r'}+F \quad \ldots 2.2$$

$$\rho\left[\frac{\partial V_\theta}{\partial t}+V_r'\frac{\partial V_\theta}{\partial r'}+\frac{V_\theta}{r'}\frac{\partial V_\theta}{\partial\theta}+\frac{V_{r'}V_\theta-V_\phi^2\operatorname{Cot}\theta}{r'}\right]$$
$$=-\frac{1}{r'}\frac{\partial P}{\partial\theta}-\frac{1}{r'^2}\frac{\partial\left(r'^2\tau_{r\theta}\right)}{\partial r'}+\frac{1}{r'\sin\theta}\frac{\partial(\tau_{r\theta}\sin\theta)}{\partial\theta}+\frac{\tau_{r'\theta}-\tau_{\phi\phi}\cot\theta}{r'}+F_\theta \quad ..(2.3)$$

$$\rho\left[\frac{\partial V_\phi}{\partial t}+V_r'\frac{\partial V_\phi}{\partial r'}+\frac{V_\theta}{r'}\frac{\partial V_\phi}{\partial\theta}+\frac{V_\phi\left(V_{r'}+V_\theta\operatorname{Cot}\theta\right)}{r'}\right]$$
$$=-\left[\frac{1}{r'^2}\frac{\partial\left(r'^2\tau_{r'\phi}\right)}{\partial r'}+\frac{1}{r'}\frac{\partial\tau_{\theta\phi}}{\partial\theta}+\frac{\tau_{r'\theta}+2\tau_{\theta\phi}\cot\theta}{r'}\right] \quad \ldots(2.4)$$

$$\rho c_p\left[\frac{\partial T}{\partial t}+V_r'\frac{\partial T}{\partial r'}+\frac{V_\theta}{r'}\frac{\partial T}{\partial\theta}\right]=-\left[\frac{1}{r'^2}\frac{\partial\left(r'^2q_{r'}\right)}{\partial r'}+\frac{1}{r'\sin\theta}\frac{\partial(q_\theta\sin\theta)}{\partial\theta}\right] \quad \ldots(2.5)$$

On substituting the following expression for the components of shear in the above equations

$$\tau_{r'r'}=-\mu\left[2\frac{\partial V_{r'}}{\partial r'}-\frac{2\nabla.V}{3}\right];\ \tau_{\theta\theta}=-\mu\left[2\left(\frac{1}{r'}\frac{\partial V_\theta}{\partial\theta}+\frac{V_r}{r'}\right)-\frac{2\nabla.V}{3}\right] \quad \ldots(2.6)$$

$$\tau_{\phi\phi} = -\mu\left[2\frac{1}{\mathrm{r}'\sin\theta}\frac{\partial V\phi}{\partial\phi} + \frac{\mathrm{V_{r'}} + \mathrm{V}_\theta\cot\theta}{\mathrm{r}'} - \frac{2\nabla.\mathrm{V}}{3}\right]$$

$$\tau_{\mathrm{r}'\theta} = \tau_{\theta\mathrm{r}'} = -\mu\mathrm{r}'\mathrm{r}'\frac{\partial(\mathrm{V}_\theta/\mathrm{r}')}{\partial\mathrm{r}'} + \frac{1}{\mathrm{r}'}\frac{\partial\mathrm{V_{r'}}}{\partial\theta} \qquad \ldots(2.7)$$

$$\tau_{\mathrm{r}'\theta} = \tau_{\theta\mathrm{r}'} = -\mu\mathrm{r}'\frac{\partial(\mathrm{V}_\theta/\mathrm{r}')}{\partial\mathrm{r}'} + \frac{1}{\mathrm{r}'}\frac{\partial\mathrm{V_{r'}}}{\partial\theta}$$

$$\tau_{r'\theta} = \tau_{\theta r'} = -\mu\mathrm{r}' + \frac{1}{\mathrm{r}'\sin}\frac{\partial\mathrm{V_{r'}}}{\partial\theta} + r'\frac{\partial(V_\phi/r')}{\partial r'} \qquad \ldots(2.8)$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} - \mu\left[\frac{\sin\theta}{\mathrm{r}'}\frac{\partial(\mathrm{V}_\phi/\sin\theta)}{\partial\theta} + \frac{1}{\mathrm{r}'\sin\theta}\frac{\partial\mathrm{V}_\theta}{\partial\phi}\right]$$

$$q_{r'} = -k\frac{\partial T}{\partial r'} \qquad q_{r'} = -k\frac{\partial T}{\partial r'} \qquad \ldots(2.9)$$

we get

$$\frac{1}{r'^2}\frac{\partial(r'^2V_{r'})}{\partial r'} + \frac{1}{r'\sin\theta}\frac{\partial(\mathrm{V}_\theta\sin\theta)}{\partial\theta} = 0 \qquad \ldots(2.10)$$

$$\rho\left(\frac{\partial\mathrm{V_{r'}}}{\partial\mathrm{t}} + \mathrm{V_{r'}}\frac{\partial\mathrm{V_{r'}}}{\partial\mathrm{r}'} + \frac{\mathrm{V}_\theta}{\mathrm{r}'}\frac{\partial\,\mathrm{Vr}'}{\partial\theta} - \frac{-(\mathrm{V}_\theta^2 + \mathrm{V}_\phi)}{\mathrm{r}}\right)$$

$$= -\frac{\partial\mathrm{p}}{\partial\mathrm{r}'} + \mu\left[\nabla^2\,\mathrm{V_{r'}} - \frac{2V_{r'}}{r'^2} - \frac{2}{r'^2}\frac{\partial\mathrm{V}_\theta}{\partial\theta} - \frac{-2\mathrm{V}_\theta}{\mathrm{r}'^2}\cot\theta\right]$$

$$+\frac{\partial\mu}{\partial\mathrm{T}}\quad 2\frac{\partial\mathrm{T}}{\partial\mathrm{r}'}\frac{\partial\mathrm{V_{r'}}}{\partial\mathrm{r}'} + \frac{1}{\mathrm{r}'}\frac{\partial\mathrm{T}}{\partial\theta}\left(\mathrm{r}'\frac{\partial(\mathrm{V}_\theta/\mathrm{r}')}{\partial\mathrm{r}'}\right) + \frac{1}{\mathrm{r}'}\frac{\partial\mathrm{V_{r'}}}{\partial\theta} + \mathrm{F_{r'}}$$

...(2.11)

$$\rho\left(\frac{\partial\mathrm{V}_\theta}{\partial\mathrm{t}} + \mathrm{V_{r'}}\frac{\partial\mathrm{V}_\theta}{\partial\mathrm{r}'} + \frac{\mathrm{V}_\theta}{\mathrm{r}'}\frac{\partial\,\mathrm{V}_\theta}{\partial\theta} + \frac{\mathrm{V_{r'}}\mathrm{V}_\theta}{\mathrm{r}'} - \frac{(\mathrm{V}_\phi^2\cot\theta)}{\mathrm{r}'}\right)$$

$$= -\frac{1}{\mathrm{r}'}\frac{\partial\mathrm{p}}{\partial\theta} + \mu\left[\upsilon^2\mathrm{V}_\theta - \frac{\mathrm{V}_\theta}{\mathrm{r}'^2\,sin\theta} + \frac{2}{\mathrm{r}'^2}\frac{\partial\mathrm{V_{r'}}}{\partial\theta}\right]$$

$$+\frac{\partial\mu}{\partial\mathrm{T}}\left(\frac{\partial\mathrm{T}}{\partial\mathrm{r}'}\left(\mathrm{r}'\frac{\partial\,\mathrm{V}_\theta/\mathrm{r}'}{\partial\mathrm{r}'}\right) + \frac{1}{\mathrm{r}'}\frac{\partial\mathrm{V_{r'}}}{\partial\theta} + \frac{2}{\mathrm{r}'^2}\frac{\partial\mathrm{T}}{\partial\theta}\left(\mathrm{Vr}' + \frac{\partial\mathrm{V}_\theta}{\partial\theta}\right)\right) + \mathrm{F}_\theta \qquad \ldots(2.12)$$

$$\rho\left[\frac{\partial\mathrm{V}_\phi}{\partial\mathrm{t}} + \mathrm{V_{r'}}\frac{\partial\mathrm{V}_\phi}{\partial_{\mathrm{r}'}} + \frac{\mathrm{V}_\theta}{\mathrm{r}'}\frac{\partial\mathrm{V}_\phi}{\partial\theta} + \frac{\mathrm{V}_\phi(\mathrm{V_{r'}} + \mathrm{V}_\theta\cot\theta)}{\mathrm{r}'}\right]$$

$$= \mu\left[\nabla^2 V_\phi - \frac{V_\phi}{r'^2 \sin^2\theta}\right] + \frac{\partial\mu}{\partial t}\left[r'\frac{\partial T}{\partial r'}\frac{\partial(V_\phi/r')}{\partial r'} + \frac{\sin\theta}{r'^2}\frac{\partial T}{\partial\theta}\frac{\partial(V_\phi/\sin\theta)}{\partial\theta}\right]$$

...(2.13)

$$\rho c_p\left[\frac{\partial T}{\partial t} + V_{r'}\frac{\partial T}{\partial r'} + \frac{V_\theta}{r'}\frac{\partial T}{\partial\theta}\right] = K\nabla^2 T - E_{r'} + \frac{\partial k}{\partial T}\left[\frac{(\partial T)^2}{\partial r'} + \frac{(\partial T)^2}{\partial\theta}\right] \quad ...(2.14)$$

Defining the following dimensionless quantities where the quantity $T_1 - T_0$ is assumed to vary:

$$t = \nu t / r_0^2, \; r = r'/r_0, \; (U_r, U_\theta, U_\phi) = (Gu_r, GU_\theta, U_\phi) r_0/G\nu, \; \Theta = (T - T_0)/(T_1 - T_0)$$
$$P = pr_0^2/\rho\nu^2, \; \Omega = \omega_0 r_0^2/G\nu, \; Pr = \mu C_p/\kappa, \; G = \alpha g r_0^2 T/\nu^2,$$
$$Nu = r_0^2 K/\kappa, \; Re = r_0 U/\nu, \; \beta = \mu_0(1+\alpha T), \; \chi = T_1 - T_0 \quad ...(2.15)$$

Then the governing equations for steady flow of motion become;

$$\frac{1}{r^2}\frac{\partial(r^2 U_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(U_\theta \sin\theta)}{\partial\theta} = 0 \quad ...(2.16)$$

$$Re\left[U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial\theta} - \frac{(U_r^2 + U_\theta^2)}{r}\right]$$
$$= -\frac{\partial P}{\partial r} + \beta\left[\nabla^2 U_r - \frac{2U_r}{r^2} - \frac{2}{r^2}\frac{\partial U_\theta}{\partial\theta} - \frac{2U_\theta \cot\theta}{r^2}\right]$$
$$+ \alpha\chi\left[\frac{\partial\Theta}{\partial r}\frac{\partial U_r}{\partial r} + \frac{1}{r}\frac{\partial\Theta}{\partial\theta}\frac{(r\partial(U_\theta/r))}{\partial r} + \frac{1}{r}\frac{\partial U_r}{\partial\theta}\right] + \Theta G\cos\theta \quad ...(2.17)$$

$$Re\left[U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial\theta} + \frac{U_r U_\theta}{r} - \frac{U_\phi^2\cot\theta}{r}\right] =$$
$$-\frac{1}{r}\frac{\partial P}{\partial\theta} + \beta\left[\nabla^2 U_\theta - \frac{U_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial U_r}{\partial\theta}\right]$$
$$+ \alpha\chi\left[\frac{\partial\Theta}{\partial r}\left(\frac{r\partial(U_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial U_r}{\partial\theta}\right) + \frac{2}{r^2}\frac{\partial\Theta}{\partial\theta}\left(U_r\frac{\partial U_\theta}{\partial\theta}\right)\right] - \Theta G\sin\theta \quad ...(2.18)$$

$$Re\left[U_r\frac{\partial U_\phi}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\phi}{\partial\theta} - \frac{U_\phi(U_r + U_\theta\cot\theta)}{r}\right]$$

$$= \beta\left(\nabla^2 U_\phi - \frac{U_\phi}{r^2 \sin^2\theta}\right) + \alpha\chi\left(r\frac{\partial\Theta}{\partial r}\frac{\partial(U_\phi/r)}{\partial r}\right) + \frac{\sin\theta}{r}\frac{\partial\Theta}{\partial\theta}\frac{\partial(U_\phi/\sin\theta)}{\partial\theta} \qquad \ldots(2.19)$$

$$\operatorname{Re}\Pr\left[U_r\frac{\partial\Theta_1}{\partial r} + \frac{U_\theta}{r}\frac{\partial\Theta_1}{\partial\theta}\right] = \alpha\chi\left[\nabla^2\Theta_1 + \left\{\left(\frac{\partial\Theta_1}{\partial r}\right)^2 + \left(\frac{\partial\Theta_1}{\partial\theta}\right)^2\right\}\Big/\beta - N_1\Theta_1\right] \qquad \ldots(2.20)$$

Following Bestman[2] we eliminate the pressure gradient to obtain the following system of differential equations:

$$\frac{1}{r^2}\frac{\partial(r^2 U_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(U_\theta\sin\theta)}{\partial\theta} = 0 \qquad \ldots(2.21)$$

$$\operatorname{Re}\frac{1}{r}\frac{\partial}{\partial r}\left[U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial\theta} + \frac{U_r U_\theta}{r} - \frac{U_\phi^2\cot\theta}{r}\right]$$
$$-\frac{1}{r}\frac{\partial}{\partial\theta}\left[U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial\theta} - \left(\frac{U_r^2 + U_\theta^2}{r}\right)\right]$$
$$+G\left(\frac{\partial\Theta}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\Theta}{\partial\theta}\cos\theta\right) = \beta\left(\nabla^2 - \omega^2 - \frac{1}{r^2\sin^2\theta}\right)\left(\frac{1}{r}\frac{\partial(rU_\theta/r)}{\partial r} - \frac{1}{r}\frac{\partial U_r}{\partial\theta}\right)$$
$$+\alpha\chi\left\{r\frac{\partial(U_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial U_r}{\partial\theta}\right\}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Theta}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2}\right\} \qquad \ldots(2.22)$$

$$Re\left[Ur\frac{\partial U_\phi}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\phi}{\partial\theta} - \frac{U_\phi(U_r + U_\theta\cot\theta)}{r}\right]$$
$$= \beta\left[\nabla^2 U_\phi - \frac{U_\phi}{r^2\sin^2\theta}\right] + \alpha\chi\left[r\frac{\partial\Theta}{\partial r}\frac{\partial(U_\phi/r)}{\partial r} + \frac{\sin\theta}{r^2}\frac{\partial\Theta}{\partial\theta}\frac{\partial(U_\phi/\sin\theta)}{\partial\theta}\right] \ldots(2.23)$$

$$= \operatorname{Re}\Pr\left[U_r\frac{\partial\Theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial\Theta}{\partial\theta}\right] = (\nabla^2 - N)\Theta + \frac{\alpha\chi}{\beta}\left[\frac{(\partial\Theta)^2}{\partial r} + \frac{1}{r}\frac{(\partial\Theta)^2}{\partial\theta}\right] \ldots 2.24$$

with;

$[U_r, U_\theta]_B = 0$, $[U_\phi]_1 = \Omega_0$, $[U_\phi]_2 = \Omega_1$, $[\Theta]_1 = 0$, $[\Theta]_2 = 1$ ... (2.25)

3 SOLUTION TECHNIQUE

For the purpose of obtaining the solutions of the equations above we assume the following expansions:

$$\Theta(r) = \Theta^0(r) + Re\Theta^1(r) + Re^2\Theta^2 + \ldots$$

$$U(r) = u(r) + Re\,U(r) + Re^2\,U^2 + \ldots \qquad \ldots(3.0)$$

Generally the order (1) equations satisfy the boundary conditions of the original problem while all subsequent higher orders must satisfy homogeneous boundary conditions.

3.1 SOLUTIONS OF THE ORDER (1) PROBLEM

The differential equations for this problem is given by the following sets of equations:

$$\frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta \cot\theta}{r} = 0 \qquad \ldots(3.1)$$

$$\beta\left[\nabla^2 - \frac{1}{r^2 \sin^2\theta}\right]\left[\frac{1}{r}\frac{\partial(u_\theta/r)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

$$+\alpha\chi\left[r\frac{\partial(u_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{r\partial\theta}\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\frac{(r\partial\Theta)}{\partial r}\right] - G\frac{\partial\Theta}{\partial r}\sin\theta = 0 \qquad \ldots(3.2)$$

$$\beta\left[\nabla^2 - \frac{1}{r^2\sin^2\theta}\right] u_\phi + \alpha\chi\left[\frac{r\partial\Theta}{\partial r}\frac{\partial(u_\phi/r)}{\partial r} = 0\right] \qquad \ldots(3.3)$$

$$\left[\nabla^2 - N\right]\Theta = 0 \qquad \ldots(3.4)$$

DIMENSIONLESS ENERGY EQUATION

The uncoupled equations of energy flow is given as

$$\frac{d^2\Theta}{dr^2} + \frac{2}{r}\frac{d\Theta}{dr} + \varepsilon_0'\Theta^2 - Nu\Theta = 0 \qquad \ldots(3.5)$$

{ Where ε_0 = $(\alpha\chi/\beta$, 1- d equation)}

On making the transformation

$r = (1+x)$ and assuming a solution

$$\Theta(X) = x + (1 - X) \sum_{m=0}^{N} a_m x^{m+1} \qquad \ldots(3.6)$$

with the equation residual $R_E(a_m, x)$ being orthogonal to the basis function the following non linear algebraic equations results in a_0, a_1 a_2:

$$126a_0^2 + 57a_1^2 + 132a_0a_1 + 60a_0a_2 + 78a_1a_2 + 33a_2^2 - (840 + 210Nu + 84\varepsilon_0)a_0 - (672 + 150Nu - 84\varepsilon_0)a_1 - (420 + 39Nu + 648\varepsilon_0)a_2 + (1260 - 126Nu + 630\varepsilon_0) = 0 \qquad \ldots(i)$$

$$792a_0^2 + 396a_1^2 + 924a_0a_1 + 484a_0a_2 + 572a_1a_2 + 248a_2^2 - (5544 + 1386Nu + 1848\varepsilon_0)a_0 - (6468 + 1089Nu - 264\varepsilon_0)a_1 - (4884 + 275Nu + 5280\varepsilon_0)a_2 + (7392 - 924Nu + 3696\varepsilon_0) = 0 \qquad \ldots(ii)$$

$$594a_0^2 + 297a_1^2 + 726a_0a_1 + 418a_0a_2 + 438a_1a_2 + 189a_2^2 - (3696 + 924Nu + 1716\varepsilon_0)a_0 - (5148 + 770Nu + 594\varepsilon_0)a_1 - (4356 + 187Nu + 4070\varepsilon_0)a_2 + (4620 - 660Nu + 2310\varepsilon_0) = 0 \qquad \ldots(ii)$$

DIMENSIONLESS VELOCITIES

Making a representation

$$u_r = (1 - x)\sum_{m-1}^{3} b_m X^{m+1}\cos\theta, \quad u_\theta = (1 - x^2)\sum_{m=1}^{3} b_{m+3} X^{m+1}\sin\theta$$

$$u_\phi = \Omega_0 + (\Omega_1 - \Omega_0)x + (1 - x^3)\sum_{m=0}^{2} c_m X^{m+1}\sin\theta$$

in the governing differential equations above with the orthogonality condition on the various equation residuals gives rise to the following linear algebraic equations in the unknown coefficients b_r and c_p respectively { $r = 1(1)6$ and $p = 0(1)2$}

$$[666666 + (607178a_2 - 87516a_0 - 34606a_1 - 152724)\varepsilon_0]b_1 +$$
$$[354354 + (15015 - 351923a_0 - 29666a_2 + 117832a_1)\varepsilon_0]b_2 +$$
$$[264264 + (15015 - 351923a_0 - 29666a_2 + 117832a_1)\varepsilon_0]b_3 +$$
$$[5225220 + (18304 + 143572a_0 - 231803a_1 + 1081925\,{}_2)\varepsilon_0]b_4 +$$
$$[777777 + (456014a_1 - 101673 - 3575a_0 + 701428a_2)\varepsilon_0]b_5 +$$
$$[8722428 - (255047a_0 - 7343 + 544258a_1 + 1784124a_2)\varepsilon_0]b_6$$
$$= 180180(1 - a_0) - (24882a_0 - 344916 - 9009a_2 + 75075a_1)G_r \qquad \ldots(i)$$

$[395538+(546910a_2-51623a_0-24310a_1-81081)\ \varepsilon_0]b_1$
$+[276705+(9152-290576a_0-24492a_2+112996\ a_1)\ \varepsilon_0]b_2$
$+[233519+(6006-22646a_0-5317a_1-222689\ a_2)\ \varepsilon_0]b_2+$
$[3413124+(12584+77935a_0-160498a_1+793364\ a_2)\ \varepsilon_0]b_4+$
$[5629264+(272415a_1-56771-71708a_0+486843a_2)\varepsilon_0]b_5+$
$[6943222-(213317a_0-2715+455363a_1+1447415a_2)\ \varepsilon_0]b_6$
$= 120120(1-a_0) - (35893a_0 - 214929 - 7670a_2+76076a_1)G_r$ ...(ii)

$[324324+(499688a_2-33176a_0-16588a_1-50908)\ \varepsilon_0]b_1+$
$[215215+(6006-247871a_0-107705a_1+19889\ a_1)\ \varepsilon_0]b_2+$
$[197769+(4160-16172a_0-3286a_1-180036\ a_2)\ \varepsilon_0]b_2$
$+[2419846+(7631+46865a_0-115219a_1+617992\ a_2)\ \varepsilon_0]b_4$ +
$[4248816+(167492a_1-34606-94926a_0+352923a_2)\varepsilon_0]b_5+$
$[5586646-(175930a_0-6566+387601a_1+1220175a_2)\ \varepsilon_0]b_6$
$= 90090(1-a_0) + (33605a_0 - 148291 - 69030a_1+14326a_2)G_r$ ...(iii)

$[240240-(50622+31746a_0+14443a_1+16159a_2)\ \varepsilon_0]b_1+$
$[156156-(19877a_0-5577+37895a_1+13507a_2)\ \varepsilon_0]b_2+$
$[156156-(13299a_0-3575+243503a_1+123695\ a_2)\ \varepsilon_0]b_2+$
$[2018016+(40183+49192a_0-136136a_1+254956\ a_2)\ \varepsilon_0]b_4+$
$[3309306+(137956a_1-35607-26884a_0+284024a_2)\ \varepsilon_0]b_5$ +
$[3860142+(27885-185263a_0-246714a_1+222400a_2)\ \varepsilon_0]b_6$
$= 30030(1-a_0) - (18018a_0 - 128700 + 39468a_1+1430a_2)G_r$ ...(iv)

$[155298-(30459+19877a_0+9867a_1+13507a_2)\ \varepsilon_0]b_1+$
$[120549-(13299a_0-3575+29289a_1+10985a_2)\ \varepsilon_0]b_2+$
$[107393-(9347a_0-2431+207974a_1+98994\ a_2)\ \varepsilon_0]b_2+$
$[1395108+(32461+28743a_0-10444a_1+178048\ a_2)\ \varepsilon_0]b_4$ +
$[2464462+(74399a_1-21164-44824a_0+202819a_2)\ \varepsilon_0]b_5$ +
$[3083080+(24830-159094a_0-208649a_1+204981a_2)\ \varepsilon_0]b_6$
$= 18018(1-a_0) + (17875a_0 - 86229 + 36608a_1+6240a_2)G_r$ ...(v)

$[32802-(7854+4290a_0+1551a_1+1331a_2)\ \varepsilon_0]b_1+$
$[15246-(2442a_0-726+3861a_1+1243a_2)\ \varepsilon_0]b_2+$
$[10626-(1529a_0-429+22561a_1+12235\ a_2)\ \varepsilon_0]b_2+$
$[246708+(3861+7260a_0-14289a_1+30349\ a_2)\ \varepsilon_0]b_4+$
$[357588+(19426a_1-5082+1793a_0+32108a_2)\ \varepsilon_0]b_5$ +

$[344022+(2266-16456a_0-22888a_1+18692a_2)\ \varepsilon_0]b_6$
$= 4620(1-a_0) - (528a_0-16632 - 803a_2+2739a_1)G_r$ ...(vi)

$[(172822a_0 + 1093338a_1 + 2185010a_2 - 2381054)\ \varepsilon_0 - 29864172]c_0+$
$[(859622a_1 - 1173646 - 1865818a_0 + 6317370a_2)\ \varepsilon_0 - 31892068]c_1$
$-[(107287a_1 + 542538 + 3384482a_0 -1292275a_2)\ \varepsilon_0 + 823752]c_2$
$= [5250960 - (1123122a_0- 2863718 +4095130a_1 + 103870a_2)\ \varepsilon_0]\ \Omega_0$
$- [2625480 + (63200 - 151584a_0- 1326a_1 + 49946a_2)\ \varepsilon_0]\ \Omega_1$ ... (vii)

$[(10556a_0 + 75502a_1 + 153618a_2 - 162188)\ \varepsilon_0 - 2445672]c_0+$
$[(60494a_1 - 64246 - 135470a_0 + 451218a_2)\ \varepsilon_0 - 2445672]c_1$
$- [(4520a_1 + 19534 + 245086a_0 -96420a_2)\ \varepsilon_0 + 38168]c_2$
$= [480480 - (73502a_0- 263406 +323752a_1 + 6188a_2)\ \varepsilon_0]\Omega_0$
$- [240240 + (66066 - 8294a_0 + 4862a_1 + 7618a_2)\ \varepsilon_0]\Omega_1$ ... (viii)

$[(10868a_0 + 47905a_1 + 95085a_2 - 83226)\ \varepsilon_0 - 1472328]c_0+$
$[(39311a_1 -17888 - 82810a_0 + 281543a_2)\ \varepsilon_0 - 1363076]c_1$
$+ [(47905a_1 + 5785 -152989a_0 +63809a_2)\ \varepsilon_0 + 48282]c_2$
$= [432432 - (25740a_0- 237666 +229515a_1 + 18161a_2)\ \varepsilon_0]\Omega_0$
$- [216216 + (69498 + 3432a_0 + 10296a_1 + 9438a_2)\ \varepsilon_0]\Omega_1$ ... (ix)

Solutions to order (Re) problems

The governing equations to the problem are given below:

$$\frac{\partial U_r}{\partial r} + \frac{2U_r}{r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_\theta \cot\theta}{r} = 0 \quad \text{...(3.7)}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial_\theta} - \frac{u_r u_\theta}{r} - \frac{u_\phi^2 \cot\theta}{r}\right)\right]$$

$$-\frac{1}{r}\frac{\partial}{\partial\theta}\left[u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_r^2 + u_\theta^2}{r}\right] + G_1 \frac{\partial \Theta_1}{\partial r}\sin\theta$$

$$= \beta\left[\nabla^2 - w^2 - \frac{1}{r^2 \sin^2\theta}\right]\left[\frac{1}{r}\frac{\partial(U_\theta/r)}{\partial r}\right) - \frac{1}{r}\frac{\partial U_r}{\partial \theta} \quad \ldots(3.8)$$

$$\left[U_r \frac{\partial u_\phi}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\phi}{\partial \theta} - \frac{u_\phi(u_r + u_\theta \cot\theta)}{r}\right]$$

$$= \beta\left[\nabla^2 - \frac{1}{r^2 \sin^2\theta}\right]U_\phi + \alpha\chi\left[r\frac{\partial \Theta^1}{\partial r}\frac{\partial(U_\phi/r)}{\partial r}\right) \quad \ldots(3.9)$$

$$Pr\, u_r \frac{\partial \Theta}{\partial r} = \frac{\alpha\chi}{\beta}\left[\nabla^2 - Nu_1\right]\Theta + \frac{2\partial\Theta\partial\Theta}{\partial r\,\partial r} \quad \ldots(3.10)$$

DIMENSIONLESS TEMPERATURE

The analysis here is as in the order(1) above . A solution of the form;

$$\Theta(x) = (1 - x^2)\sum_{m=0}^{2} a_{m+3} x^{m+1}$$

therefore yields the following system of linear system in the unknown coefficients a_3, a_4 and a_5:

$$[786744 - (132704a_0 - 40040 + 98930a_1 + 65390\,a_2)\,\varepsilon_0 + 63206\,Nu\,]a_3 +$$
$$[564564 - (69524a_0 + 40040 + 74802a_1 + 61330\,a_2)\,\varepsilon_0 + 37908\,Nu\,]a_4$$
$$+[385528 - (31590a_0 + 61074 - 126860a_1 + 1368720\,a_2)\,\varepsilon_0 + 64792\,Nu\,]a_5$$
$$= 0 \quad \ldots(i)$$

$$[604604 - (3830324a_0 - 86372 + 71682a_1 + 49922\,a_2)\,\varepsilon_0 + 37908\,Nu\,]a_3 +$$
$$[511940 - (1290302a_0 - 14742 + 54688a_1 + 45564\,a_2)\,\varepsilon_0 + 24752\,Nu\,]a_4 +$$
$$[387270 - (28998a_0 + 14742 - 1065474a_1 + 1141386\,a_2)\,\varepsilon_0 + 53174\,Nu\,]a_5$$
$$= 0 \quad \ldots(ii)$$

$$[943800 - (155948a_0 - 181116 + 117016a_1 + 83760\,a_2)\,\varepsilon_0 + 570024\,Nu\,]a_3 +$$
$$[876096 - (96092a_0 - 69524 + 89048a_1 + 73976\,a_2)\,\varepsilon_0 + 34276\,Nu\,]a_4 +$$
$$[707616 - (56376a_0 - 14256 - 1811124a_1 + 1936380\,a_2)\,\varepsilon_0 + 180612\,Nu\,]a_5$$
$$= 0 \quad \ldots(iii)$$

In the same way the assumption of the following

$$U_r=(1-x^2)\sum_{m=0}^{2}b_{m+7}x^m\cos\theta,\quad U_\theta=(1-x^6)\sum_{m=0}^{2}b_{m+10}x^m\sin\theta,$$

$$U_\phi=(1-x^3)\sum_{m=0}^{2}c_{m+3}x^{m+1}\sin\theta$$

gives rise to the linear system below:

$$\begin{aligned}
&[143480714-(22237564a_0+39045362a_1+17672350a_2-1892644)\varepsilon_0]\,b_7+\\
&[110293586+(24453684a_0+57067266a_1+23653494a_2+1512660)\varepsilon_0]\,b_8+\\
&[83809762+(51360128a_0-121334627a_1-134010213a_2+2428314)\varepsilon_0]\,b_9\\
&+[-29016416+(152560652a_0+282910600a_1+537922228a_2+21883080)\varepsilon_0]b_{10}+\\
&[-670855224+(190658536a_0+364296077a_1+9684705a_2-22231512)\varepsilon_0]\,b_{11}+\\
&[-608973456+(215074057a_1-17559912-45862294a_0+152631033a_2)\varepsilon_0]\,b_{12}\\
&=(76410988a_4-6385846a_3+4277880a_5)b_1+(13762690a_4-20555550a_3+\\
&554888670a_5)b_2-(12240034a_3-58768446a_5+11074837a_4)b_3-\\
&(283145880a_3-31054920a_5+151046802a_4)b_4+\\
&(112870106a_3-2095250618a_5+43993450a_4)b_5\\
&-(135084346a_3-692671506a_5+27356886a_4)b_6]\,\varepsilon_0+\\
&[(52854802b_4-4550832b_1-54960048b_2-12252240b_3-130832b_5+602446\,b_6)b_1+\\
&(218348\,b_2+31556148\,b_3-130832\,b_4-49810\,b_6+602446b_5)\,b_2\\
&+(54453720\,b_3+602446b_4-49810b_5+2612084\,b_6)b_3-(4883658b_4-\\
&3503156b_6+76988444b_5)b_4+(584392b_5+317152b_6)b_5+59806b_6^2]\beta_0\\
&-(401308+4890288a_0-5891860a_1+1646552a_2)G_r \qquad \ldots(i)
\end{aligned}$$

$$\begin{aligned}
&[1991963610-(294417084a_0+645819120a_1+283955760a_2-4050420)\varepsilon_0]\,b_7\\
&+[1729363318+(441805214a_0+992587075a_1+411880005a_2+16207494)\varepsilon_0]\,b_8\\
&+[1360527034+(896865702a_0-152270161a_1-290813829a_2+33070678)\varepsilon_0]\,b_9\\
&+[-1055880540+(2876042388a_0+4478791644a_1+12243130620a_2+356807764)\varepsilon_0]\,b_{10}\\
&+[-8802799104+(3251400688a_0-397459252-165935189a_2+6078225229a_1)\varepsilon_0]\,b_{11}\\
&+[-8538410684+(3792869813a_1-341366426-791372306a_0+2564536785a_2)\varepsilon_0]\,b_{12}\\
&=(1139149294a_4-185142954a_3+30182412a_5)b_1+(210421903a_4-343281170a_3\\
&+724315416a_5)b_2-(177944576a_3-872614710a_5+170236618a_4)b_3+\\
&(4574305974a_3-454897430a_5+2346306238a_4)b_4-\\
&(214530236a_3-3607080810a_5+292598398a_4)b_5-(2613315860a_3-\\
&301963206a_5+267448142a_4)b_6]\,\varepsilon_0+\\
&[(802311328b_4-113070672b_1-1175132140b_2-534096004\,b_3-1486446b_5+\\
&9030434b_6)\,b_1+(2099500\,b_2+550140060b_3-1486446\,b_4-632434\,b_6+9030434b_5)\,b_2\\
&+(977107140b_3+9797882b_4-632434b_5+44825940b_6)b_3-(82543942b_4-\\
&31770280b_6+1095615516b_5)b_4-(10534968b_5+8156316b_6)\,b_5-477071b_6^2]\beta_0-\\
&(401308+4890288a_0-5891860a_1+1646552a_2)G_r \qquad \ldots(ii)
\end{aligned}$$

$$[\,1528719594 - (757630738a_0 +1641596789a_1 +238881015a_2 +12060174)\,\varepsilon_0\,]\,b_7$$
$$+[1435700762 + (414897376a_0 +917344299a_1 +383552677a_2 + 9262348)\,\varepsilon_0\,]\,b_8$$
$$+[114980002 + (827436795a_0 - 120972069a_1 - 275636283a_2 + 214582561)\,\varepsilon_0\,]\,b_9$$
$$+[-576455515+(1274541204a_0+3807354235a_1+1276755481 0a_2 +321729964)\varepsilon_0]\,b_{10}$$
$$+[-7256989580+(2921497893a_0-363081393-157564415a_2 + 5411339631a_1)\varepsilon_0]\,b_{11}$$
$$+[-7251771192+(3343105731a_1-332452671-747708558a_0+2292241491a_2)\,\varepsilon_0\,]\,b_{12}$$
$$=[(170236618a_4 - 225267952a_3 +60313854a_5)b_2- (130907378a_3 - 913101943a_4 +12138150a_5)b_1 +(139230024a_3- 80586051a_5 + 139016160a_4)b_3- (3928851508a_3 - 333874422a_5 + 2081404989a_4)b_4- (211718222a_3-2860199240a_5+203154718a_4)b_5- (246689046a_3-1219176690a_5+7545202a_4)b_6]\,\varepsilon_0$$
$$+[(667792164b_4 - 122031338b_1 - 742719120b_2 - 492894124\, b_3 - 946390b_5 + 7303676b_6)\,b_1+(160208\, b_2+ 507390364b_3 - 946390\, b_4 - 432980\, b_6+7303676b_5\,)\,b_2 +(931658616b_3+8437406b_4-439280b_5+40871451b_6\,)b_3- (72118148b_4 - 23903292b_6+894945144b_5)b_4-(9292710b_5+8469060b_6)b_5-1232454b_6^2]\beta_0$$
$$-(387483720+111892244a_0 - 116799384a_1+18327020a_2)\,G_r \qquad ...(iii)$$

$$[\,3893472362 - (582468484a_0 -120379516-506520848a_1 +414468432a_2)\,\varepsilon_0\,]\,b_7$$
$$+[2567447022 - (2477797606a_0 +219116417a_1 +1108911231a_2 -31550640)\,\varepsilon_0\,]\,b_8$$
$$+[2849677190 - (258465892a_0 + 234588649a_1 +1349642713a_2 -54057280)\,\varepsilon_0\,]\,b_9$$
$$+[-20291247600-(1310091876a_0-4383549280a_1 -9678205712a_2 +146501172)\varepsilon_0]\,b_{10}$$
$$+[1585206480+ (602183112a_0 + 4595019617a_1+2012618415a_2- 660047916)\varepsilon_0]\,b_{11}$$
$$+[-10732240900+(346185244a_0+4043199563a_1+1071559079a_2+321822342\,)\varepsilon_0]\,b_{12}$$
$$=[(1749961028a_4-1639407172a_3 +829080436a_5)b_2+(317250277a_4 -490842428a_3 +1080174852a_5)b_1 +(277871732a_3-1382212608a_5+256012498a_4)b_3- 8214935120a_3+25975246560a_4+22371432320a_5)b_4 -(1634370430a_3-579743574a_5+639124066a_4)b_5- (58196546a_3+138787192a_4+1343877978a_5)b_6]\,\varepsilon_0$$
$$+[(154825528b_1+112146892b_2+42527472b_3-34079084b_4-2779738b_5+13834090b_6)b_1 - (218444900b_3 - 295122516\, b_2 - 13834090b_5 +2779738\, b_4 +1085926\, b_6)\, b_2 +(155598144b_3+14232026b_4-1085926b_5+41592710b_6)b_3- (32517056b_4-5782180b_6+1352327356b_5)b_4-(14196496b_5+8912216b_6)b_5 +683791b_6^2]\beta_0 -(88692780-35734900a_0-170589220a_1+24372288a_2)\,G_r \qquad ...(iv)$$

$$[\,2992726138 - (294338272a_0 - 78109152 - 423103837a_1 +350445861a_2)\,\varepsilon_0\,]\,b_7$$
$$+[2090313694 - (174434212a_0 +190751089a_1 +949118191a_2 -17777920)\,\varepsilon_0\,]\,b_8$$
$$+[2454117178 - (196571397a_0 + 184222575a_1 +1177691303a_2 -39164073)\,\varepsilon_0\,]\,b_9$$
$$+[-16797847560 -(1106904204a_0-3676668435a_1-8694551943a_2+135247206)\varepsilon_0\,]\,b_{10}$$

$+[3017476336+(682295225a_0 + 4240480135a_1+1879084953a_2- 605451549)\varepsilon_0]\, b_{11}$
$+[-9378749200+(450205418a_0+3598189813a_1 + 808850349a_2 +232462701)\, \varepsilon_0]\, b_{12}$
$=[(1383530125a_4-1428121224a_3 +704902470a_5)b_2+(256012498a_4 - 432203716a_3$
$+891753258a_5)b_1 +(214195740a_3-117186517a_5+20790316a_4)b_3-(16460615530a_3$
$+23395410900a_4+21741099330a_5)b_4-(1529100650a_3-02087974a_5+690464333a_4)b_5-$
$(73209778a_3+138187953a_4+1369570638a_5)b_6]\varepsilon_0$
$+[(99146788b_1+63236940b_2+27210812b_3 -28645578b_4-1685414b_5+11007194b_6)\, b_1$
$- (185007940b_3 - 63236940\, b_2 -11007194b_5 +1685414b_4 +733210\, b_6)\, b_2$
$+(129364084b_3+12216506b_4-733210b_5+38284867b_6)b_3-(30413680b_4-$
$8509432b_6+1053424418b_5)b_4(13180984b_5+10861844b_6)b_5-1016690b_6^{\,2}]\beta_0$
$+(308167840a_0+157697644a_1+11373446a_2 - 622123840)\, G_r$ ...(v)

$[\, 2402658110 - (210713572a_0 - 54057280 - 368284391a_1 +295508311a_2)\, \varepsilon_0]\, b_7$
$+[1773724138 - (127565031a_0 +146916075a_1 +823641959a_2 -10065003)\, \varepsilon_0]\, b_8$
$+[2121698426 - (153813835a_0 + 147187160a_1 +1042286478a_2 -29375007)\, \varepsilon_0]\, b_9$
$+[-14365092960 -(968369086a_0-3153236675a_1- 7927325233a_2+115247046)\, \varepsilon_0]\, b_{10}$
$+[3703674332+ (685286433a_0+3918218301a_1+1752373693a_2- 547250363)\, \varepsilon_0]\, b_{11}$
$+[-6708187386+(482169086a_0+3221138607a_1+ 605257587a_2 +178186617)\, \varepsilon_0]\, b_{12}$
$=[(961452250a_4 - 1265621236a_3 +588210132a_5)b_2+(207903016a_4 - 382115232a_3$
$+746193432a_5)b_1+(168687700a_3-1008445536a_5+170448894a_4)b_3-$
$(1506185390a_3+21289939070a_4+19613881160a_5)b_4-(1432278240a_3-$
$429124964a_5+716363295a_4)b_5- (93860410a_3+135554435a_4+136978342a_5)b_6]\, \varepsilon_0$
$+[(67570308b_1+31492500b_2+16241732b_3-224964670b_4-1085926b_5+ 8966480b_6)\, b_1$
$- (159979316b_3 - 232208576\, b_2 - 8966480b_5 +1085926b_4 +513893b_6)\, b_2$
$+(108753777b_3+10553702b_4-513893b_5+35442885b_6)b_3-$
$(25983412b_4 - 37963482b_6+835766376b_5)b_4- (11545635b_5+10928610b_6)b_5-$
$1794246b_6^{\,2}]\beta_0+(260180376a_0+149080004a_1+28191440a_2 - 465145840)\, G_r$ **(vi)**

$[\, 8269252200 - (869102560a_0+261232045a_1 + 1155335375a_2 +1831219430)\varepsilon_0]\, c_3$
$+[\, 4869877460- (352755520a_0 + 236607285a_1 + 199394485a_2 +244927640)\varepsilon_0]\, c_4$
$+[\, 5014991670-(7092709375a_0 +323399710a_1 +1933154650a_2 +318474675)\varepsilon_0]\, c_5$
$= [(419690050\Omega_1-839380100\Omega_0-2923029690c_0-1550216365\, c_1+4662852130\, c_1)a_3$
$+(79338490\Omega_1-158676980\Omega_0- 3686672885c_0 - 5778902605\, c_1+4836993650\, c_2)a_4$
$-(460875780\Omega_1-921751560\Omega_0+1532763630c_0+2641542480c_1+293923290c_2)a_5]\varepsilon_0$ ...**(vii)**

$[\, 9628630000-(1047595590a_0+300488515a_1+1359001695a_2+1831219430)\varepsilon_0]\, c_3$
$+[\, 5504378660- (288645720a_0 + 272366945a_1 + 230785115a_2 -201209620)\varepsilon_0]\, c_4$

$+[\,5569921820-(1724944545a_0+339308460a_1+2061675250a_2+629239530)\varepsilon_0\,]\,c_5$
$=[\,(185049930\Omega_1-370099860\Omega_0-1721425270c_0-2692534460c_1+5065398860\,c_2\,)a_3$
$+(46902830\Omega_1-93805660\Omega_0-4142292505c_0-1849260785c_1+559634930\,c_2\,)a_4$
$-(542504340\Omega_1-1085008680\Omega_0+2617396110c_0+2443210950c_1+399219060c_2\,)a_5]\varepsilon_0$ ...(**viii**)

$[\,596700000-(70843250a_0+193689505a_1+86286050a_2+117387550)\varepsilon_0\,]\,c_3$
$+[\,316771200-(17338980a_0+17492830a_1+14816430a_2-3840980)\varepsilon_0\,]\,c_4$
$+[\,643915800-(36983160a_0+18262095a_1+115540035a_2+2367930)\varepsilon_0\,]\,c_5$
$=(60288880\Omega_1-120577760\Omega_0-213419360c_0-142034150c_1+291455990c_2\,)a_3$
$-(2320500\Omega_1-4641000\Omega_0-27258085\,c_2+234953600c_0+445267570c_1)a_4$
$-(35530170\Omega_1-71060340\Omega_0+162885840c_0+185816370c_1+28487460c_2)a_5\,]\varepsilon_0$ ...(**ix**)

Note: The values of a_k, b_r and c_k { k=0(1)5,r=0(1)11}are obtained from a turbo basic program developed by the author of this article in Aiyesimi[7].This program computes the values of the a_k's from the set of nonlinear algebraic equations and substitutes same into the linear algebraic systems defining the , b_r's and c_k's and thereafter solves the linear systems for the unknown coefficients. Finally these values so obtained are substituted in the final part of the program to obtain the non-dimensional temperatures and velocities.

4 ERROR ANALYSIS

Considering the differential equation ;

$$Lu = f(u,x) \qquad ...4.1$$

where L is an invertible differential operator and f(u,x) satisfies a Lipschitz condition with respect to u. Denoting our trial function (approximate solution) by u_T and the corresponding equation residual by R_E we thus have that ;

$$R_E = Lu_T - f(u_T,x)$$

consequently we have

$$L^{-1}R_E = L^{-1}Lu_T - L^{-1}f(u_T,x)$$

ie, $$u_T = L^{-1}R_E + L^{-1}f(u_T,x) \qquad ...4.2$$

Similarly $$u = L^{-1}f(u,x) \qquad .4.3$$

ie

$$|u-u_T| = |L^{-1}f(u,x) - L^{-1}f(u_T,x)| + |L^{-1}R_E|$$

$$||L^{-1}|\,|f(u,x)-f(u_T,x)| + |L^{-1}||\;R_E|$$

$$< |L^{-1}|\, K\, |u - u_T| + L^{-1}\, |R_E|$$

Hence ,

$$|u - u_T| < \frac{L^{-1}\|R_E\|}{1-K\|L^{-1}\|} \qquad \text{....4.4}$$

This thus gives an error bound for the approximate method.

DISCUSSION OF RESULTS

Assigning the following numerical values to the parameters

$\Omega = 1.0$, $\Omega = .50$, $\alpha = 8.40$ E- 05 , Pr =0.71, Re = 0.001, T_0 = 780°C, T_1 = 790°C,

	Θ(x)		$V_\Phi(x)$	
	$\varepsilon_0 = 0.005$	$\varepsilon_0 = 0.00$	$\varepsilon_0 = 0.005$	$\varepsilon_0 = 0.00$
x=0.0	0.0000000000	0.0000000000	1.0000000000	1.0000000000
= 0.1	0.2702283561	0.2699992657	0.6620181203	0.6620272398
=0.2	0.4900972843	0.4897768795	0.6229365468	0.6229595805
=0.3	0.6643067466	0.6639992594	0.5963224769	0.5963394046
=0.4	0.7975527644	0.7973327637	0.5870413780	0.5870497823
=0.5	0.8945374489	0.8944440484	0.5966821909	0.5966710448
=0.6	0.9599550302	0.9599996965	0.6208844185	0.6208440854
=0.7	0.9985166788	0.9986662865	0.6458974481	0.6458730396
=0.8	1.0149097442	1.0151107311	0.6443706251	0.6442758441
=0.9	1.0138376951	1.0139995813	0.5703765750	0.5702936649
=1.0	1.0000000000	1.0000000000	0.3536647558	0.3536647358

All the physical constants used in this write - up are adopted from Weltty et`al [8].

CONCLUSION

In most cited work in fluid dynamics the mathematical models describing the flow field have always been based on the governing equations that are entirely formulated on the assumption that the properties are always constant. The results based on such models are bound to be at variance with reality as shown in [1] due to the non consideration of the variability of such properties like the viscosity thermal conductivity and difussibility at certain temperature range.

The effect of variable fluid properties is clearly manifested in the results obtained above.

Results obtained from the numerical simulations above indicate an increase in the dimensionless energy and a decrease in the velocities in the inner half of the annulus when the variability of the viscosity and the thermal conductivity of the fluid is considered in the flow analysis as compared with the constant property concept. The reverse is the case in the outer half of the annulus . From the tables above the flow profiles obtained by setting the variability parameter $\alpha = 0$ correspond with analysis based on constant properties . Therefore results so obtained for the temperature profiles without considering the variability are in general lower in the interior and higher in the outer region of the annulus than the actual results . In the case of the dimensionless velocity there is a general overestimation of results in the inner part and underestimation in the outer region of the annulus .

APPENDIX

Definitions of symbols used with their physical significance

a_k Unknown coefficients for the dimensionless Temperature.
b_k, c_k Unknown coefficients for the dimensionless velocities.
c_p coefficient of thermal conductivity of fluid.
κ coefficient of thermal diffusibility of fluid .
μ coefficient of static viscosity of fluid
ν coefficient of dynamic viscosity of fluid
F Body force
Θ Dimensionless Temperature
Gr Grashoff number
Nu Nusselt number
Pr Prandtl number
Re Reynolds number
Ω_0 Angular velocity of inner sphere
Ω_1 Angular velocity of outer sphere
T Dimensional Temperature
U Dimensionless Velocity
V Dimensional Velocity

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