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ANALYTICAL SOLUTIONS OF THE BLOCH EQUATIONS FOR MRI FLOW DEPENDENT MAGNETIZATION AND SIGNALS WITH SPATIALLY VARYING Rf $B_1(x)$ MAGNETIC FIELD

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ABSTRACT

Nuclear Magnetic Resonance (NMR) has been established as a powerful tool to study the structure of complex molecules, molecular motion, rate processes and molecular interactions. Despite the tremendous theoretical and technical advanced improvement of this new medical diagnostic in many research laboratories around the world, there is a common belief, however, that basic NMR properties of biological materials are still far from being fully understood and a great deal of further research is needed to clarify the quantitative and qualitative information in various NMR studies. An ideal approach to extract this useful information would be to find generalized solution to the Bloch equation, but this is obviously a difficult task. In this study, we present a generalized analytical solution of the Bloch equations for magnetic resonance imaging flow dependent magnetization and signal with spatially varying rF $B_1(x)$ magnetic field.

INTRODUCTION

The adult body contains some six quarts of blood and about 60,000 miles (96,500 kilometres) of blood vessels, including tiny capillaries. At its normal rates of about 70 beats per minutes, the heart will pump some six quarts (6 litres) of blood every minute. Think of it! Your heart pushes your entire blood supply through your body in less than 60 seconds! Under ordinary conditions, it pumps up to 10 tons of blood through your vessels daily. Yet, at this rate, it is not even working very hard.

If yours is a physically fit heart, one trained by regular exercise, it may be capable of pumping as many as 30 quarts of blood or more in a minute. At that rate it is pushing your entire blood supply through your body about every 10 seconds! Yes, your heart pumps so steadily and powerfully that every day it can push your blood through several thousand complete circuits of your body!

Such a marvelously designed pumping and flow system in human body may make you to wonder: How does blood flow through these 96,500 km of blood vessels in human body in less than 60 seconds? What is the danger posed by any obstacle to the normal flow of blood in the body or any obstruction in the blood vessels?

Knowledge of blood flow and circulation through the vessel of various organs of the body is important for a number of clinical applications. Life depends so much on blood such that its importance cannot be over emphasized. It has been investigated that any obstacle to the normal flow of blood in the body causes a malfunctioning in the body system. An accurate measurement of blood velocity as a function of time serves as an indicator of heart and vessel diseases. Similarly, a profile, of distribution of blood velocity across vessels indicates the presence or absence of obstruction in the blood vessels.

It is from the importance of blood flow mentioned above that necessitated the search for simple, fast and accurate methods of blood flow measurement and accurate determination of vessel cross section. In this study our main concern is the Nuclear Magnetic Resonance (NMR) methods to achieve this purpose. The said technique is noninvasive.

Blood flows with almost no resistance in all the large vessels of the circulation. Some resistance is however posed by arterioles and capillaries. The small resistance to blood flow in these small vessels is overcome by the high pressure (systolic pressure) of about 120mmHg in the systemic system and 22mmHg in the pulmonary system, with which the heart pumps blood into the arteries. It is important to know that the flow is not steady but pulsates and a portion of the energy from the heart is stored elastically in the arterial system to maintain flow while the heart is being refilled.

The amount of blood that passes a point at a given period of time is called blood flow rates. The blood flow rate, Q, at a point depends on the blood velocity V, and the cross section, Ω_p of the vessel at that point. Thus $Q = V\Omega_p$.

Noninvasive measurements of blood flow in the human vascular system are of considerably interest for clinical diagnosis, because such measurements make it possible to assess the functional state of the heart and circulatory system. Various magnetic resonance (MR) vascular imaging have been developed[1-12] both to provide detailed anatomic images and information about the distribution of velocities within an image element.

A general analytical solution of the Bloch equations for Magnetic resonance flow dependant magnetizations and signals with spatially varying magnetic field gradient was chosen by the author for this study because the Bloch equations are coupled nonlinear equations describing the motion of a macroscopic magnetization 'M' under the influence of applied magnetic fields. The basic Physics of extracting the relevant flow parameters from the analytical solution of Bloch equations can provide a clue to design a practical MRI sequence by which velocity quantification can be made accurately.

In this contribution, generalized analytical solution of Bloch equations for NMR fluid flow with spatially varying magnetic field gradient would be presented. The basic physics of extracting the relevant flow parameters from the solution of the Bloch equation would make it possible to design the magnetic resonance sequence without the restrictions imposed by other methods [9-20].

THEORY

For this investigation, we assumed that resonance condition exists at Lamor frequency

$$f_o = \gamma B \tag{1}$$

In the following, γ denotes the gyromagnetic ratio of blood spins; $\omega/2\pi$ is the rF excitation frequency; f_o/γ is the off- resonance field in the rotating frame of reference.

The x,y,z components (in the rotating frame) of magnetization of blood bolus are given by the Bloch equations (Slichter 1963), which may be written as follows:

$$\frac{dM_x}{dt} = V \bullet gradM_x + \frac{\partial M_x}{\partial t} = \frac{-M_x}{T_2} \tag{2}$$

$$\frac{dM_y}{dt} = V \bullet gradM_y + \frac{\partial M_y}{\partial t} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \tag{3}$$

$$\frac{dM_z}{dt} = V \bullet gradM_z + \frac{\partial M_z}{\partial t} = -\gamma M_y B_1(x) + \frac{M_0 - M_z}{T_1} \tag{4}$$

Two reasonable initial boundary conditions which may conform to the real-time experimental arrangements are chosen. These are

1. $Mo \neq Mz$

a situation which holds good in general and in particular when the rF $B_1(x)$ field is strong say of the order of 1.0G or more.

2.before entering signal detector coil, blood bolus has magnetization

$M_x = 0$, $M_y = 0$

If $B_1(x)$ is large; $B_1(x) \approx 1G$ or more so that M_z of the blood bolus changes appreciably from M_o.

For steady flow:

$$\frac{\partial M_y}{\partial t} = 0 \tag{5}$$

From equations (3) and (4) we can write

$$\frac{d^2 M_y}{dx^2} + \frac{1}{V}\left(\frac{1}{T_1} + \frac{1}{T_2}\right)\frac{dM_y}{dx} + \frac{\gamma^2}{V^2} B_1^2(x) M_y = \frac{\gamma M_0 B_1(x)}{V^2 T_1} \tag{6}$$

We define

$$R = \frac{1}{V}\left(\frac{1}{T_1} + \frac{1}{T_2}\right) = \frac{1}{VT_0}$$

where

$$\frac{1}{T_0} = \left(\frac{1}{T_1} + \frac{1}{T_2}\right); Q(x) = \frac{\gamma M_0 B_1(x)}{V^2 T_1}$$

and

$$\gamma^2 B_1(x) >> \frac{1}{T_1 T_2} \qquad (7)$$

Then equation (4) becomes

$$\frac{d^2 M_y}{dx^2} + \frac{R dM_y}{dx} + S(x) M_y = Q(x) \qquad (8)$$

S(x) is a function that defines the relationship between the magnetic field and velocity. Employing a transformation Z = E(x) such that

$$\frac{dM_y}{dx} = \frac{dM_y}{dZ}\frac{dZ}{dx}; \quad \text{and} \quad \frac{d^2 M_y}{dx^2} = \frac{d^2 M_y}{dZ^2}\left(\frac{dZ}{dx}\right)^2 + \frac{dM_y}{dZ}\frac{d^2 Z}{dx^2}$$

equation (8) becomes

$$\frac{d^2 M_y}{dZ^2} + \frac{P dM_y}{dZ} + \frac{S(x) M_y}{\left(\frac{dZ}{dx}\right)^2} = \frac{Q(x)}{\left(\frac{dZ}{dx}\right)^2} \qquad (9)$$

where

$$P = \frac{\left(\frac{dZ^2}{dx^2} + \frac{RdZ}{dx}\right)}{\left(\frac{dZ}{dx}\right)^2} \tag{10}$$

we choose Z= E(x) in such a way that

$$\frac{dZ}{dx} = \sqrt{\frac{S(x)}{a^2}} = \left(\frac{\gamma^2 B_1^2(x)}{V^2}\right)^{\frac{1}{2}} = \pm\frac{\gamma B_1(x)}{V} \tag{11}$$

Equation (9) then gives

$$\frac{d^2 M_y}{dZ^2} + P\frac{dM_y}{dZ} + M_y = \frac{M_0}{T_1 \gamma B_1(x)} \tag{12}$$

For equation (9) to have solution, P must be treated as a constant. We found it useful in this analysis for P to be zero and a^2 consistently equal to unity. From equation (10)

$$\frac{dZ}{dx} = Ce^{-Rx} = \pm\frac{\gamma B_1(x)}{V} \tag{13}$$

Since we are seeking for finite solutions we must choose C to be negative such that

$$B_1(x) = \pm\frac{CV}{\gamma}e^{-Rx} = \pm\frac{VRZ}{\gamma} = \frac{Z}{\gamma T_0} \tag{14}$$

$$B(0) = \frac{CV}{\gamma} \tag{15}$$

$$B_1(x) = B_1(0)e^{-Rx} = (-)^n B_1(0)\left(1 + Rx + \frac{R^2x^2}{2!} + \frac{R^3x^3}{3!} + \ldots\ldots\ldots\right) \tag{16}$$

Equation (15) can be experimentally verified. Equation (14) is true only for x>0. However, for equation (12) to be valid, the field gradient must be designed within the limit of equation (15).

MAGNETIC FIELD GRADIENT

The function, e^{-Rx}, is always decreasing, there are no turning point and no point of inflection, we can neglect n>1 terms in equation (16) , then we can write

$$B_1(x) = B_1(0) - B_1(0)Rx \tag{17}$$

The main magnetic field is defined as

$$\bar{B} = B_o + B_1(x) = B_o - B_1(0) + \Delta B = B_c\bar{z} + G \bullet x\bar{z} \tag{18}$$

Where $\Delta B = B_1(0)Rx$; $G = \frac{d\Delta B}{dx}$; $B_c = B_o + B_1(0)$ is the static magnetic field.

For more general cases, the magnetic field gradients may be present in all the three spatial directions:

$G(t) = [G_x(t), G_y(t), G_z(t)]$

The constant velocity V of the moving spins along the direction of G can be determined from equations (18) and (7) for human blood $T_1 \cong 1.0s$ and $T_2 \cong 0.15s$. The velocity obtained are similar for T_2 in the range $0.075s<T_2<0.5s$ and $0.75s<T_1<1.5s$.

THE IMAGING EQUATION

Let us write equation (12) as

$$\left(D_z^2 + (a+b)D_z + ab\right)M_y = Q(z) \tag{19}$$

Where

$P = (a+b)$; $ab = 1$; $Q(z) = M_o/\gamma T_1 B_1(x) = T_o M_o/ZT_1$

We can confidently neglect the complementary solution of equation (12), since it is independent of magnetization M_o. Therefore at the initial instant, when the blood bolus is about to enter the excitor coil,

$M_y = 0$; $X=0$; and $Z = C/R = C/VT_o$

Then a particular integral of equation (12) is

$$M_y = \frac{1}{(D+a)} \bullet \frac{1}{(D+b)} Q(z) = \frac{T_o M_o}{T_1} e^{-az} \int e^{(a-b)z} \int \frac{e^{bz}}{z} (dz)^2 \tag{20}$$

$$M_y(z) = \frac{M_o}{T_1 VR}(2+P)^{-1}\frac{1}{z} \tag{21}$$

If C/R is chosen so that

$$Z = \frac{C}{R}e^{-Rx} = CVT_o e^{-\frac{x}{VT_o}} \tag{22}$$

Then we can write

$$M_y(x) = \frac{M_o}{VCT_1}e^{\frac{x}{VT_o}} \tag{23}$$

If we now turn to equations (1) and (18), so as to express the phase in terms of the 'static' fields applied, the MR signal induced by spins at the position x(t) which is excited by a radio-frequency pulse $\gamma B_1 t_\omega$ is given by

$$S(t) = A\int M_y(r,t)e^{2\pi i\gamma G.rt}d^3r \tag{24}$$

Where A= $c'\Omega_p$, c' is the instrumental factor and t_ω is the pulse width. In equation (24) , it is possible to adjust and maintain Bo so that the off-resonance phase term is kept negligibly small over the time duration of the measurement. The detected signal can finally be expressed as

$$S(t) = A\int M_y(r,t)e^{-2\pi i q.r}d^3r \tag{25}$$

Where

$q \equiv -\gamma Gt$

We observe the fundamental relation of NMR imaging in equation (25); the phase-coherently detected signal is proportional to the spatial Fourier transform of the magnetization,

$$\tilde{m}(q) \equiv \int M_y(r,)e^{2\pi i\gamma q.r}d^3r \tag{26}$$

That is, if we define $q \equiv -\gamma Gt$ and use the solution

$$\tilde{S}(q) \equiv S(t) \tag{27}$$

with

$$t=(q/\gamma G) \tag{28}$$

then equation (25) gives

$$\tilde{S}(q) = A\tilde{M}_y q \tag{29}$$

as the detected signal pair. By obtaining a sufficiently dense set of all q values as suggested by equation (29), the Fourier transform can be inverted numerically to produce an image of $M_y(r)$. One accomplishes this by repeating many cycles of S(t) measurement, with M_y excited identically, but with G changed from cycle to cycle in effective direction. The simplification of making P=0 allows one to measure the MRI signal at its peak value.

DISCUSSION AND CONCLUSION

This study presents some applied mathematics to provide analytical solutions to the Bloch equations for magnetization and signal with spatially varying magnetic field. Three orthogonal field gradient modulations provide spatial-position encoding of the NMR signals. The quantities measured are the spatial Fourier-transform components of the precessing magnetization and the quantities imaged is the spatial distribution of precessing magnetization. The precessing magnetization is proportional to the physical density and velocity of precessing spins, T_1-relaxation factors, T_2-decay factors and, in principle, other physical effects. These sensitizations may be generated in a variety of ways, in each case, a specific intracycle method generates the appropriate sensitization and thereby modulates the physical spin density in producing that distribution of precessing magnetization. It should be noted in this analysis that the magnetic field $B_1(x)$ is defined only for P = 0, however, when P>0, the measured NMR signal can be very small and may not yield good results.

For better appreciation of the mathematical analysis presented here, mathematical modeling of the physical properties discussed in this article will be required to determine the reproducibility and clinical feasibility of this method. This will be the focus of our next research activities and would be reported later.

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