J. Nig. Ass. Math. Phys.
Vol. 3 (1999)

GENERALIZED SOLUTIONS TO NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT
We consider the non –linear parabolic equation

$$u_t - \Delta u + f(u) = 0 \qquad \in Q = \Omega \times]0, T[\qquad T > 0$$
$$u(x,t) = u_0(x) \qquad \in \Omega$$
$$u(x,t) = 0 \qquad \in \partial\Omega \times]0, T[$$

Where $u_0(x)$ is a distribution with compact support or more generally, a generalized function associated with such a distribution. Ω is a bounded open set in R^n with smooth boundary $\partial\Omega$ and $n \in Z_+$ is a positive integer. This equation arises often in chemical flow problems, gas dynamics and other physical processes and is known to have no weak solution in the classical distributional sense [1]. We show that the algebra of generalized functions G(Q) [2] provides a suitable setting for the construction of solutions to such a problem. We obtain existence, uniqueness as well as consistence results for the solutions to the problem using non classical methods which involves the use of classical estimates and the induction hypothesis over the order of the differential operators which defines the elements of G(Q).

1.0 INTRODUCTION

J. F Colombeau [2] and [3] recently constructed successfully a space of distributions which generalizes the classical concept of a differentiable function. Colombeau's theory provides a suitable setting for the construction of solutions to many important partial differential equations of mathematical physics for which classical concepts produces ambiguous results [3,4,5,6].

Using non-classical methods, we obtain existence, uniqueness as well as regularity results for the non-linear parabolic system

$$u(x,0) = u_o(x) \qquad \in \Omega$$
$$u(x,t) = 0 \qquad \in \partial\Omega \times]0, T[\qquad (1.1)$$

Where $u_o(x)$ is a distribution with compact support or more generally, a generalized function associated with such a distribution. Ω is a bounded open set in R^n with smooth boundary $\partial\Omega$ and $n \in Z_+$ is a positive integer. Equation

(1.1) models many physical processes which includes time –dependent irreversible processes such as heat conduction, chemical reactions and biological flow problems.

In the realm of the classical theory of differential equations it can be shown, imposing a Liptchizt condition on the non-linearity f that (1.1) has a unique solution if $u_0(x) \in C^\infty(\Omega)$. On the other hand, if $u_0(x)$ is a distribution with compact support, the system (1.1) has no solution in the classical distributional sense [1] which makes the classical theory of distributions inadequate in describing solutions to (1.1). The algebra of generalized functions G(Q) provides a suitable setting for obtaining solutions to this problem. Using this new concept, we obtain existence, uniqueness as well as consistence results for the solutions to the problem (1.1). Our technique involves the use of classical estimates and the induction hypothesis over the order of the differential operators defining the elements of G(Q) and differ substantially from those considered in [7] where classical techniques where employed.

The rest of this paper is organized as follows, in section 2, we give a simplified definition of the algebra G(Q) that allows restriction to subspaces. More sophisticated definitions are given in [8, 9,10]. In section 3, under some mild assumptions on the non-linearity *f*, we obtain existence as well as uniqueness results for (1.1). Finally in section 4, we show that the generalized solutions obtained in section 3 are consistent with the classical distributional solutions.

2.0 THE CONCEPT OF GENERALIZED FUNCTIONS USED.

In order to solve nonlinear problems involving general multiplication of distributions and their physical applications, J. F Colombeau introduced a new theory of 'Generalized Functions' which are more general than distributions by an ingenious construction of the algebra $G(\Omega)$ of generalized functions which has $C^\infty(\Omega)$ as a subalgebra and contains the space $D'(\Omega)$ of distributions. This resulted from the idea of constructing a space of functions for which derivatives as well as nonlinear operations are preserved, thus providing a means of generalization of the classical concept of a differentiable function. We give here a simplified construction of the algebra $G(\Omega)$ which permits restriction to subspaces and which depends on a given system of coordinates. The definitions given here can be found in [9]. Other more sophisticated construction can be found in [2] and [8].

Definition 2.1 Let N_0 denote the set of natural numbers including zero and *D*(R) the set of C^∞ functions defined on R, vanishing outside a variable compact subset of R. Set

$$A_p(R) = \left\{X \in D(R) : \int X(x)dx = 1; \int x^k X(x)dx = 0\right\}$$

whenever $1 \le k \le p ; p \in N_o$

and

$$A_p(R^n) = \left\{ \prod_{j=1}^{n} X(x_j); X \in A_p(R) \right\}$$

We denote by $E[\Omega]$ the set of all functions of the form $F : A_o \times \Omega \to R$ where for every θ,φ the map

$$\underset{\lambda}{R} \times \underset{\varepsilon}{]0,1[} \times \underset{x}{\Omega} \to \underset{F([\lambda\theta+(1-\lambda)\varphi]_\varepsilon, x)}{R}$$

is C^∞ in the variable λ, ε, x.

Given $F \in E[\Omega]$ and $\varphi \in A_0(R^n)$ we write $F(\varphi, x)$ for the value of $F(\varphi)$ at the point $x \in \Omega$. Observe that $E[\Omega]$ as defined above with pointwise multiplication is an algebra. Also if T is a distribution then the convolution $(T*\varphi)(x)$, whenever it is defined is also a C^∞ function for $\varphi \in A_o(R^n)$ so that one has an imbedding

$$T \to [x \to [\varphi \to (T*\varphi)(x)]] \qquad (2.1)$$

Which makes $D'(\Omega)$ a subspace of $E[\Omega]$. But $E[\Omega]$ does not contain $C^\infty(\Omega)$ as a subalgebra, since the convolution product of *F* (*F* a continuous function on R) and $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(x\varepsilon^{-1})$ namely $F*\varphi_\varepsilon$ is not equal to *F* in general, no matter the smallness of ε. In other words, the inclusion of $C^\infty(\Omega)$ into $E[\Omega]$ and the inclusion as a subspace of *C* do not give the same result. In order to make the two inclusions above coherent, we define a subspace $E_M[\Omega]$ of moderate elements of $E[\Omega]$ as follows

Definition 2.2 A subset $E_M[\Omega]$ of $E[\Omega]$ is called moderate if for every compact subset K of Ω and every differential operator $\partial^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$ $(\alpha_i \in N_o)$, $\alpha = \sum_{i=1}^{n} \alpha_i$; there exist an element q∈ N_o such that for every $\varphi \in A_q(R^n)$ there exist c>0 and η> 0 such that

$$\sup_{x\in K}\left|\partial^{\alpha}F(\varphi_{\varepsilon},x)\right| \le C\varepsilon^{-q} \quad \text{whenever} \quad 0<\varepsilon<\eta \tag{2.2}$$

$E_M[\Omega]$ thus defined is a differential algebra for componentwise operations.

Definition 2.3 An ideal $\tilde{N}[\Omega]$ of $E_M[\Omega]$ is called Null if for every differential operator ∂^{α}, there exist $N \in N_o$ such that for all $q \ge N$ and all $\varphi \in A_q(R^n)$ there exist $c > 0$ and $\eta > 0$ such that

$$\sup_{x\in K}\left|\partial^{\alpha}F(\varphi_{\varepsilon},x)\right| \le c\varepsilon^{q-N} \quad \text{whenever } 0<\varepsilon<\eta \tag{2.3}$$

Observe that the elements in $\tilde{N}[\Omega]$ have a faster decay than any power of ε as $\varepsilon \to 0$.

Definition 2.4 Set

$$G(\Omega) = \frac{E_M[\Omega]}{\tilde{N}[\Omega]}$$

That is the quotient algebra of $E_M[\Omega]$ with respect to $\tilde{N}[\Omega]$. The algebra $G(\Omega)$ has the following properties: the space of distributions, $D'(\Omega)$ is contained in $G(\Omega)$ through the formula (2.1). Furthermore, if $T \in C^{\infty}(\Omega)$ then (2.1) defines the same element in $G(\Omega)$ as the constant imbedding $T \to [x \to [\varphi \to (T^*\varphi)(x)]\ ]$. This makes $C^{\infty}(\Omega)$ a subalgebra of $G(\Omega)$. Finally, the derivative on $G(\Omega)$, defined on representatives by $(\partial x_i F)(\varphi,x) = \partial x_i(F(\varphi,x))$ extends differentiation on $D'(\Omega)$.

3.0 GENERAL EXISTENCE AND UNIQUENESS RESULT

Lemma 1. Let $v \in C^{\infty}(Q)$ satisfy the equation

$$\begin{aligned}
&(\partial_t - \Delta)v(x,t) + a_o(x,t)v(x,t) = g(x,t) && \text{in } Q\\
&v(x,t) = v_o(x) && \text{on } \Omega\\
&v(x,t) = h(x,t) && \text{in } \partial\Omega\times[0,T]
\end{aligned}$$

Where $T > 0$ is finite and where $a_o(x,t) \ge 0$ in Q, then there is a polynomial P_λ in one variable with coefficient independent of a_o, g, v_o and h such that

$$\sup_{(x,t)\in Q}|v(x,t)| \le \left(\sup_{x\in\Omega}|v_o(x)| + \sup_{(x,t)\in Q}|g(x,t)| + \sup_{(x,t)\in(\partial\Omega\times[0,T]}|h(x,t)|\right)P_\lambda\left(\sup_{x,t\in Q}|a_o(x,t)|\right)$$

Proof. See [7]

Definition 3.1 A smooth real valued function f : Q → Q is said to satisfy the bounded gradient condition if for every $\dot{u} \in G(Q)$ there exist $\alpha > 0$ such that

$$\sup_{u\in Q}|\partial_u f(u)| < \alpha \tag{3.1}$$

The following theorem gives a general existence, uniqueness result for the solution to problem (1.1).

Theorem 1. Let f : Q → Q be a smooth function which together with it's derivatives satisfy the bounded gradient condition (3.1), then for every $u_o \in G(\Omega)$ with compact support, there exist a unique element $u \in G(Q)$ satisfying

$$\begin{aligned} &u_t - \Delta u + f = 0 && \text{in } G(\Omega) \\ &u|_{\Omega\times\{Q\}} = u_o && \text{in } G(\partial\Omega\times[0,T]) \\ &u|_{\partial\Omega\times[0,T]} = 0 && \text{in } G(Q) \end{aligned} \tag{3.2}$$

representative $u(\varphi_\varepsilon, x, t)$ be the classical C^∞ solution to (3.2) with initial data $u_0(\varphi_\varepsilon, x)$ at t = 0 defined in such a manner that given any C^∞ function $\lambda \in D(\Omega)$ then $\lambda u_0(\varphi_\varepsilon, x)$ is still a representative of u_0 vanishing outside some neighbourhood of the support of u_0. If λ is chosen such that λ is identical to 1 on a neighborhood of supp u_0, then $u_0(\varphi_\varepsilon, x)$ is such that $u_0(\varphi_\varepsilon, x) = 0$ whenever x is outside supp u_0. This solution exists and is unique. Hence we conclude that there is an element $u(\varphi_\varepsilon, x, t) \in E[Q]$ satisfying (3.2). In order to prove that $u(\varphi_\varepsilon, x, t)$ so defined is a generalized solution to (3.2) we have to show that $u(\varphi_\varepsilon, x, t)$ is moderate, that is, we show that for every compact subset K ⊂⊂ Q and every differential operator ∂^α, there is p ∈ N such that for every $\varphi \in A(R^{n+1})$ there is c>0, η>0 such that

$$\sup_{(x,t)\in K}\left|\partial^{\alpha}u(\varphi_{\varepsilon},x,t)\right|\le c\varepsilon^{-p} \qquad 0<\varepsilon<\eta \tag{3.3}$$

The proof is by induction over the order of the differential operator ∂^{α}. Take $\alpha = 0$, then from the maximum principle we have

$$\sup_{(x,t)\in Q}\left|u(\varphi_{\varepsilon},x,t)\right|\le \sup_{x\in\Omega}\left|u_0(\varphi_{\varepsilon},x)\right| \tag{3.4}$$

But by definition, $u_0(\varphi_{\varepsilon},x)\in E_M[\Omega]$, therefore there exist $p\in N_o$ such that for all $\varphi\in A_p(R^n)$ there is $\eta>0$, $c_1>0$ such that

$$\sup_{x\in L}\left|u_0(\varphi_{\varepsilon},x)\right|\le c_1\varepsilon^{-N} \qquad 0<\varepsilon<\eta \tag{3.5}$$

Where L is any compact subset of Ω. Therefore an inequality of type (3.3) holds when ∂^{α} is the identity operator. Assume now that (3.3) holds for all differential operators of order $\le l$, we wish to derive (3.3) for $\alpha = l+1$. Differentiating (3.2) $l+1$ times, we see that $\partial^{l+1}u(\varphi_{\varepsilon},x,t)$ satisfies an equation of the form

$$(\partial_t-\Delta)\partial^{l+1}u(\varphi_{\varepsilon},x,t)+\partial_u f(u(\varphi_{\varepsilon},x,t)).\partial^{l+1}u(\varphi_{\varepsilon},x,t)+$$

or

$$(\partial_t-\Delta)\partial^{l+1}u(\varphi_{\varepsilon},x,t)+\partial_u f(u(\varphi_{\varepsilon},x,t)).\partial^{l+1}u(\varphi_{\varepsilon},x,t)+P(f,u(\varphi_{\varepsilon},x,t),\partial_u f,\partial u(\varphi_{\varepsilon},x,t)....)$$

terms involving products of various powers of the derivative of f and u of order $\le l$ (3.6)

where P is a smooth function which is polynomially bounded for $u\in Q$ and contains only derivatives of f and u of order $\le l$. From the maximum principle $\partial^{l+1}u(\varphi_{\varepsilon},x,t)$ achieves it's maxima on $\partial\Omega\times[0,T]$. Since $u(\varphi_{\varepsilon},x,t)$ depends continuously on the initial data $u_o(\varphi_{\varepsilon},x)$ there exist c>0 such that

$$\sup_{x\in\partial\Omega}\sup_{t\in[0,T]}\left|\partial^{l+1}u(\varphi_{\varepsilon},x,t)\right|\le c\sup_{x\in\partial\Omega}\left|\partial^{l+1}u_0(\varphi_{\varepsilon},x)\right| \tag{3.7}$$

but $u_o(\varphi_{\varepsilon},x)$ is moderate, therefore $\partial^{l+1}u_o(\varphi_{\varepsilon},x)$ defines an element in $E_M[\Omega]$, which implies there exist $c_o>0$ such that for ε sufficiently small and $p_1\in N_o$

$$\sup_{x\in\partial\Omega}\sup_{t\in[0,T]}\left|\partial^{l+1}u(\varphi_\varepsilon,x,t)\right|\le c_o\varepsilon^{-p} \tag{3.8}$$

By the induction hypothesis and the bounded gradient condition (3.1), there is $p_2 \in N_o$ such that for $\varphi\in A_{P_2}(R)$ there exist $c_2>0$ such that

$$\sup_{(x,t)\in\Omega}\left|P(f,u(\varphi_\varepsilon,x,t),f'(u),\ldots)\right|<c_2\varepsilon^{-p_2} \tag{3.9}$$

for ε sufficiently small. Finally from (3.1), there is $p_3 \in N_o$, $c_2>0$ and $\varphi\in A_{P_3}(R^{n+1})$ such that

$$\sup_{(x,t)\in Q}\left|\partial_u f(u(\varphi_\varepsilon,x,t))\right|<c_3\varepsilon^{-p_3} \tag{3.10}$$

thus from lemma1, we have

$$\sup_{(x,t)\in Q}\left|\partial^{l+1}u(\varphi_\varepsilon,x,t)\right|\le\left(c_0\varepsilon^{-p_1}+c_2\varepsilon^{-p_2}\right)P_\lambda c_3\varepsilon^{-p_3} \tag{3.11}$$

which defines an element in $E_M[Q]$.

The solution thus obtained is unique since if there are two solutions $u_1,u_2\in G(Q)$ satisfying the same initial and boundary conditions, then their difference u_1-u_2 satisfies

$$\begin{aligned}&(\partial_t-\Delta)(u_1-u_2)+f(u_1)-f(u_2)=0 && \text{in } G(Q)\\ &u_1-u_2|_{\Omega\times\{0\}}=0 && \text{in } G(\Omega)\\ &u_1-u_2|_{\partial\Omega\times[0,T]}=0 && \text{in } G(\partial\Omega\times[0,T])\end{aligned} \tag{3.12}$$

Replacing $f(u_1)-f(u_2)$ by $\int_0^1 f'(\xi u_1-(1-\xi)u_2)d\xi.(u_1-u_2);\ 0\le\xi\le 1$ (3.12) becomes

$$\begin{aligned}&(\partial_t-\Delta)(u_1-u_2)+\theta(u_1,u_2)(u_1-u_2)=0 && \text{in } G(Q)\\ &u_1-u_2|_{\Omega\times\{0\}}=0 && \text{in } G(\Omega)\\ &u_1-u_2|_{\partial\Omega\times[0,T]}=0 && \text{in } G(\partial\Omega\times[0,T])\end{aligned} \tag{3.13}$$

Where

$$\theta(u_1,u_2)=\int_0^1 f'(\xi u_1-(1-\xi)u_2)d\xi \qquad (3.14)$$

Since f satisfies the bounded gradient condition, it implies that θ is uniformly bounded Let $u_i(\varphi_\varepsilon,x,t)$ $i = 1, 2$ be representatives of u_1 and u_2 respectively and let $u(\varphi_\varepsilon,x,t)=u_1(\varphi_\varepsilon,x,t)-u_2(\varphi_\varepsilon,x,t)$. Then from (3.13), $u(\varphi_\varepsilon,x,t)$ satisfies the equation

$$\begin{aligned}(\partial_t-\Delta)u(\varphi_\varepsilon,x,t)+\theta(u_1(\varphi_\varepsilon,x,t),u_2(\varphi_\varepsilon,x,t)).u(\varphi_\varepsilon,x,t)&=g(\varphi_\varepsilon,x,t)\\ u(\varphi_\varepsilon,x,t)\big|_{\partial\Omega\times[0,T]}&=h(\varphi_\varepsilon,x,t)\\ u(\varphi_\varepsilon,x,0)\big|_{\Omega\times\{0\}}&=v_o(\varphi_\varepsilon,x)\end{aligned} \qquad (3.15)$$

Where $g(\varphi_\varepsilon,x,t)\in \tilde{N}[Q]$, $v_o(\varphi_\varepsilon,x)\in \tilde{N}[\Omega]$ and $h(\varphi_\varepsilon,x,t)\in \tilde{N}[\partial\Omega\times[0,T]]$.

Proving that $u_1=u_2$ amounts to showing that $u(\varphi_\varepsilon,x,t))\in \tilde{N}[Q]$. In other words we have to show that for every differential operator ∂^α, there is $p\in N_o$ such that for every $q > p$ and every $\varphi\in A_p(R^{n+1})$ there exist $c>0$ such that

$$\sup_{(x,t)\in Q}|\partial^{n+1}u(\varphi_\varepsilon,x,t)|\le C\varepsilon^{q-p} \qquad (3.16)$$

The proof again is by the induction hypothesis over the order of the differential operator ∂^α. Since g, v_o and h are null and by assumption θ is uniformly bounded, then from lemma1, there exist c_1, $c_2>0$ such that for all $p_1\in N_o$, there exist $q > p_1$ such that

$$\sup_{(x,t)\in Q}|u(\varphi_\varepsilon,x,t)|\le c_1\varepsilon^{-q-p_1}P_\lambda(c_2)$$

for all $\varphi\in A_q(R^{n+1})$. Thus an inequality of type (3.16) holds for the case $\alpha = 0$. Assume now that (3.16) holds for all differential operators of order $\le l$. Differentiating (3.14) $l+1$ times, we see that $\partial^{l+1}u(\varphi_\varepsilon,x,t)$ satisfies an equation of the form

$$\begin{aligned}&(\partial_t-\Delta)\partial^{l+1}u(\varphi_\varepsilon,x,t)-\theta(u_1(\varphi_\varepsilon,x,t),u_2(\varphi_\varepsilon,x,t)\partial^{l+1}u(\varphi_\varepsilon,x,t)\\ &=\partial^{l+1}g(\varphi_\varepsilon,x,t)+\text{terms of order}\le l+\partial^{l+1}\theta(u_1(\varphi_\varepsilon,x,t),u_2(\varphi_\varepsilon,x,t))u(\varphi_\varepsilon,x,t)\end{aligned}$$

where $\partial^r \theta(u_1(\varphi_\varepsilon, x, t), u_2(\varphi_\varepsilon, x, t))$ can be calculated from (3.15) and the equations for u_1 and u_2. From (3.15) and (3.1) we see that there exist a constant c>0 such that

$$\sup_{(x,t)\in\Omega\times\{0\}} \left|\partial^{l+1} u(\varphi_\varepsilon, x, 0)\right| + \sup_{x\in\Omega}\left|\partial^{l+1}\Phi(u_0(\varphi_\varepsilon, x))v_0(\varphi_\varepsilon, x)\right| \leq$$

$$C\left\{\sup_{(x,t)\in\Omega\times\{0\}} \left|\partial^{l+1} g(\varphi_\varepsilon, x, 0)\right| + \sup_{x\in\Omega}\left|\sum_{j=0}^{l} F_j(u_0(\varphi_\varepsilon, x), \ldots, \partial^j u_0(\varphi_0, x))\partial^{l-j} v_0(\varphi_\varepsilon, x)\right|\right. \quad (3.17)$$

where Φ and F_j $j = 0,\ldots l$ can be calculated from the condition (3.14) and are polynomially bounded while $u_0(\varphi_\varepsilon, x)$ is a representative of the initial data for $u_1(x,0) - u_2(x,0)$. Now since $u_0(\varphi_\varepsilon, x)$ is moderate and both $g(\varphi_\varepsilon, x, 0); v_0(\varphi_\varepsilon, x)$ are null, we have that there is $N \in N_0$ such that for $q \geq N$ and $\varphi \in A_q(R^{n+1})$

$$\sup_{(x,\epsilon)\in\Omega\times\{0\}} \left|\partial^{l+1} u(\varphi_\varepsilon, x, 0)\right| \leq c_1 \varepsilon^{q-N}$$

for ε small enough. Also from the maximum principle, there is $c_2 > 0$ such that

$$\sup_{(x,\epsilon)\in\partial\Omega\times[0,T]} \left|\partial^{l+1} u(\varphi_\varepsilon, x, t)\right| \leq c_2 \sup_{x\in\partial\Omega}\left|\partial^{l+1} u(\varphi_\varepsilon, x, 0)\right|$$

Therefore since $v_0(\varphi_\varepsilon, x)$ is $\in \tilde{N}$ there is $N_1 \in N_0$ such that for $q \geq N_2$ and $\varphi \in A_q(R)$

$$\sup_{(x,\epsilon)\in\Omega\times[0,T]} \left|\partial^{l+1} u(\varphi_\varepsilon, x, t)\right| \leq c_3 \varepsilon^{q-N_2}$$

For small ε. Finally since θ is uniformly bounded together with it's derivatives then by lemma 1, there exist $N_3 \in N_0$ such that for $q \geq N_3$ and $\varphi \in A_{N_3}(R^{n+1})$ there exist $c_4 > 0$ such that

$$\sup_{(x,\epsilon)\in Q} \left|\partial^l u(\varphi_\varepsilon, x, t)\right| \leq c_4\, \epsilon^{q-N_3}$$

Which completes the proof of the theorem.

4. CONSISTENCE OF THE SOLUTION

Uniqueness as defined in G(Q) does not imply uniqueness in the distributional sence since the equality in G(Q) is too strong a way for defining equality in $D'(Q)$. In order to make the concept of equality in G(Q) coherent with the equality in $D'(Q)$, J.F Colombeau [2] introduced the idea of association or weak equality in G(Q). Thus G(Q) is endowed with two kinds of 'equalities', the strong equality denoted by '=' and the weak equality or association denoted by '$\approx$'. Two elements u_1 and $u_2 \in G(Q)$ are said to be associated if for every test function $\psi \in D(\Omega)$, $\int_Q (U_1 - U_2)\psi dxdt \to 0$ as $\varepsilon \to 0$ where U_1, U_2 are representatives of u_1 and u_2 respectively. We may also have association of elements in G(Q) with their corresponding elements in $D'(\Omega)$.

Definition 4.1 An element $u \in G(\Omega)$ with representative $u(\varphi_\varepsilon, x, t)$ is said to admit an element $v \in D'(Q)$ as an associated distribution iff for every test function $\psi \in C_0^\infty(Q)$ there is $N \in N_0$ such that

$$\int_Q u(\varphi_\varepsilon, x, t)\psi(x,t)dxdt \to v(\psi)$$

as ε tends to zero, for all $\psi \in A_N(R^n)$.

The following theorem show that the generalized solutions obtained in sec. 3 are consistent with the distributional solutions using the concept of weak equality in G(Q).

Theorem 3. Assume f:Q→Q satisfies the condition (3.1) and u is the classical C^∞ solution to (1.1) with initial data u(x,0) = δ, then if v_0 is the equivalence class of δ in G(Ω) and if u_0 admits an element $v_0 \in D'(\Omega)$ as an associated distribution, then the solution to (1.1) converges to an element in $D'(Q)$.

Proof Let u be the classical C_0^∞ solution to (1.1) then by theorem 1 u is moderate, hence there is a representative $u(\varphi_\varepsilon, x, t)$ of u in G(Q) satisfying

$$\begin{aligned}
&(\partial_t - \Delta)u(\varphi_\varepsilon, x, t) + f(u(\varphi_\varepsilon, x, t)) = g(\varphi_\varepsilon, x, t) && \text{in } G(Q)\\
&u(\varphi_\varepsilon, x, 0) - u_0(\varphi_\varepsilon, x) = e(\varphi_\varepsilon, x) && \text{in } G(\Omega)\\
&u(\varphi_\varepsilon, x, t) = h(\varphi_\varepsilon, x, t) && \text{in } G(\partial\Omega \times [0,T]) \qquad (4.1)
\end{aligned}$$

where $u_0(\varphi_\varepsilon, x, t)$ is a representative of the initial data for $u(x,t)$ and *g, e, h* are Null or more explicitly $g \in \tilde{N}[Q]$, $e \in \tilde{N}[\Omega]$ and $h \in \tilde{N}[\partial\Omega \times [0,T]]$. Now let $\psi(x,t) \in C^2(Q)$ vanishing on $\partial\Omega \times [0,T]$ be a solution of the parabolic equation

$$\begin{aligned} &\partial_t \psi - \Delta\psi + f(u)\psi = \chi && \text{in } Q \\ &\psi(x,0) = \gamma(x) && \text{in } \Omega \times \{0\} \\ &\psi(x,T) = 0 && \text{in } \partial\Omega \times [0,T] \end{aligned} \tag{4.2}$$

where $\chi \in C_0^\infty(Q)$ and $\gamma \in C_0^\infty(\Omega)$. Multiply (4.2) by $\psi(x,t)$ and integrating we see that

$$\int_Q [\partial_t u(\varphi_\varepsilon, x, t) - \Delta u(\varphi_\varepsilon, x, t) + f(u(\varphi_\varepsilon, x, t))]\psi(x,t)dxdt = \int_Q g(\varphi_\varepsilon, u, t)\psi(x,t)dxdt$$

integration by parts and the use of (4.2) yields

$$\begin{aligned} \int_Q u(\varphi_\varepsilon, x, t)[\chi(x,t) - g(\varphi_\varepsilon, x, t)]dxdt &= \int_\Omega u(\varphi_\varepsilon, x, T)\psi(\varphi_\varepsilon, x, T)dx \\ &- \int_\Omega u_0(\varphi_\varepsilon, x)\gamma(x)dx + \int_{\partial\Omega\times(0,T)} h(\varphi_\varepsilon, x, t)\frac{\partial\psi}{\partial\eta}(\varphi_\varepsilon, x, t)d\eta dt \end{aligned} \tag{4.3}$$

Observe that ψ depends on ε through (4.2). Thus taking limits as $\varepsilon \to 0$ in (4.3) we notice that since $g(\varphi_\varepsilon, x, t) \in N[Q]$ and $u(\varphi_\varepsilon, x, t) \in E_M[Q]$, thus

$$\int_Q u(\varphi_t, x, t)\, g(\varphi_\varepsilon, x, t)dxdt \to 0 \text{ as } \epsilon \to 0.$$

Similarly since $\frac{\partial\psi}{\partial\eta} \in E_M[\partial Q \times [0,T]]$ and $h(\varphi_\varepsilon, x, t) \in N[\partial\Omega \times [0,T]]$, then

$$\int_{\partial\Omega\times(0,T)} h(\varphi_\varepsilon, x, t)\frac{\partial\psi}{\partial\eta}(\varphi_\varepsilon, x, t)\partial\eta\partial t \to 0 \text{ as } \varepsilon \to 0.$$

Finally we observe that since the sequence of smooth functions $\{U(\varphi_\varepsilon, x, T)\}_{0<\varepsilon<1}$, $\{\psi(\varphi_\varepsilon, x, T)\}_{0<\varepsilon<1}$ satisfy $\int_\Omega U(\varphi_\varepsilon, x, T)\psi(\varphi_\varepsilon, x, T)d\Omega \underset{\varepsilon\to 0}{\to} V(\psi)$

where $V \in D'(Q)$, it follows from (4.3) and the foregoing that

$$\lim_{\epsilon\to 0}\int_Q U(\varphi_\varepsilon, x, t)\,\chi(x,t)dxdt$$

converges to an element in $D'(Q)$.

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