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DISCRETE EIGENVALUES OF A MODIFIED PÖSCHL – TELLER POTENTIAL HOLE USING SEMI-CLASSICAL APPROXIMATION

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ABSTRACT

The use of a modified Pöschl-Teller potential hole is addressed using both semi-classical path integral and Exact methods. The exact results of eigenvalues of a modified Pöschl-Teller potential hole is obtained on solving the one-dimensional Schrödinger equation for this potential hole analytically, which shows the accuracy of the semi-classical method. The approximate quantization rules (the well known Bohr-Sommerfeld quantization conditions) are derived from the W.K.B.J. formulae (solutions). The semi-classical method is then tested by the quantization of the quantum mechanical systems of the modified Pöschl-Teller potential hole which practically gives the same results as using exact method.

1. INTRODUCTION

The W.K.B.J. formula (Wentzel 1926, Kramers, 1926, Brillioiun 1926 and Jeffreys 1923) [1], [2] Cocolicchio & Viggiano, 1997 [3] provides us with rather simple and interestingly good approximate solution to the Schrödinger equation, for this reason it is widely used in many approximate calculations of quantum mechanical systems. Different tunnelling effects, as well as other effects caused by potential barriers Giler et al 1986 [4], Farina 1988 [5], Méndez and Dominguez-Adame 1994 [6] are examples in which the use of the W.K.B.J. formulae give correct results.

Other forms of the applications of this approximation method is the well known Bohr-Sommerfeld quantization conditions. For examples, Oyewumi and Bangudu 1998 [7] used W.K.B.J. method with the derivation of Bohr-Sommerfeld quantization conditions, obtained the discrete eigenvalues (quartic) oscillator which agrees with Giler's result of 1988 [1]. Giler used kth generalised Bohr-Sommerfeld quantization conditions, Ghatak et al 1997 [8] applied the JWKB formula to a triangular potential barrier and compare the resits with the exact results, also Birx and Hourk 1977 [9] obtained the eigenvalues for the upper and lower bounds of anharmonic oscillator potential. Although WKB fails to predict the individual energy levels (and also their wavefuncitons) within a vanishing fraction of mean energy level spacing in the limit when the quantum number goes to infinity Robnik and Salasnich 1997 [10].

The exact results of eigenvalues of a modified Pöschl-Teller potential hole is as obtained in §2 on solving the one-dimensional Schrödinger equation for this potential hole analytically. While §3 contains the derivation of Bohr-Sommerfeld quantization conditions through W.K.B.J. formulae. The application of this semi-classical approximation method to modified Pöschl-Teller potential hole and the comparison of the results of the exact method with the semi-classical method is contained in §4. While §5 contains discussion of results and we conclude with §6.

2. EXACT EIGENVALUES OF A MODIFIED PÖSCHL-TELLER POTENTIAL HOLE

The non-trivial potential, modified Pöschl-Teller potential hole for which the exact solution of the discrete energy is known is used in which the semi-classical approximation method gives an accurate results. Chebotarev 1996 [11] obtained the total delay time and tunnelling time for non-rectangular potential barriers and used modified Pöschl-Teller potential hole as an example which its exact result is known. Grypeos and Liolios 1997 [12] used the Hypervirial theorems (HVT) in conjunction with the Hellman-Feynman theorem (HFT) which provide a very powerful scheme for the treatment of the Even-Power potentials and one of the examples used is modified Pöschl-Teller potential hole and with the method its energy compared favourably with the exact analytic expression. Infact this potential also has been used rather extensively in studieis of hypernuclei, Lalazissis et al 1988 [13], Lalazissis 1994 [14] and Lalazissis 1993 [15].

An interesting feature of this potential is that the corresponding Schrödinger eigenvalue problem can be solved exactly.

The modified Pöschl-Teller potential hole [8] & [9] is given by

$$V(x) = \frac{-\hbar^2 \alpha^2 \lambda(\lambda - 1)}{2\mu \operatorname{Cosh}^2 \alpha x} \tag{2.1}$$

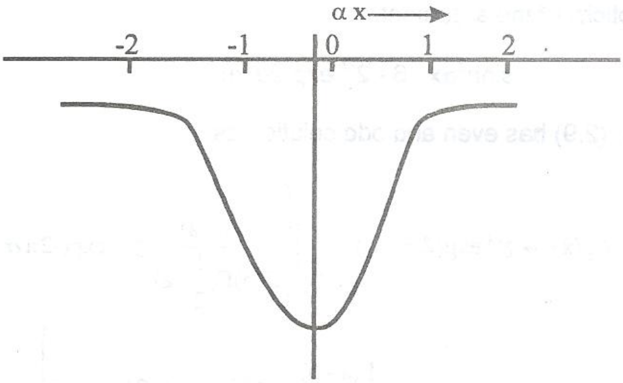
With $\lambda > 1$

The one-dimensional Schrödinger equation for the modified Pöschl-Teller potential hole is:

$$\Psi''(x) + \left[k^2 + \frac{\alpha\lambda(\lambda - 1)}{\operatorname{Cosh}^2 \alpha x}\right]\Psi(x) = 0 \tag{2.2}$$

$$\text{with } k^2 = \frac{2\mu}{\hbar^2} \tag{2.3}$$

Fig. 1: Modified Pöschl-Teller Potential Hole



We introduce the new variable;

$$y = \mathrm{Cosh}^2 \alpha x \tag{2.4}$$

We obtain

$$y(1-y)\Psi''(y) + \left[\frac{1}{2} - y\right]\Psi'(y) - \left[\frac{k^2}{4\alpha^2} + \frac{\lambda(\lambda-1)}{}\right]\Psi(y) = 0 \tag{2.5}$$

On substituting,

$$\Psi(y) = y^{\lambda/2}\, U(y) \tag{2.6}$$

To split off a fitting power of y, we arrive at the hypergeometric differential equation [18]

$$y(1-y)\, U''(y) + \left[\left(\lambda + \frac{1}{2}\right) - (\lambda+1)\, y\right] U'(y) - \frac{1}{4}\left(\lambda^2 + \frac{k^2}{\alpha^2}\right) U(y) = 0 \tag{2.7}$$

$$a = \frac{1}{2}\left(\lambda + \frac{ik}{\alpha}\right); b = \frac{1}{2}\left(\lambda - \frac{ik}{\alpha}\right) \tag{2.8}$$

The complete solution of equation (2.7) may be written as:

$$U(x) = AF(a, b, \frac{1}{2}; 1 - y) + B(1 - y)^{1/2} F\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; 1 - y\right) \qquad (2.9)$$

This can be simplified further using [1] with the boundary conditions and the asymptoticity of the argument,

$$- \sinh^2 \alpha x \quad 6 - 2^{-2} \exp(2\alpha|x|)$$

Equation (2.9) has even and odd solution as:

$$\Psi_e(x) \to 2^{-\lambda} \exp(\lambda\alpha |x|) \Gamma\left(\frac{1}{2}\right) \left\{ \frac{\Gamma(b-a)}{\Gamma(b)\Gamma(\frac{1}{2} - a)} 2^{2a} \exp(-2a\alpha |x|) + \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(\frac{1}{2} - b)} 2^{2b} x \exp(-2b\alpha |x|) \right\} \qquad (2.10a)$$

$$\Psi_o(x) \to \pm 2^{-(\lambda+1)} \exp[(\lambda+1)\alpha |x|] \Gamma\left(\frac{3}{2}\right) \left\{ \frac{\Gamma(b-a)}{\Gamma(b+\frac{1}{2})\Gamma(1-a)} 2^{2a+1} \exp(-(2a+1)\alpha |x|] + \frac{\Gamma(a-b)}{\Gamma\left(a+\frac{1}{2}\right)\Gamma(1-b)} 2^{2b+1} \exp[-(2b+1)\alpha |x|] \right\} \qquad (2.10b)$$

Then , equation (2.2) now, compose a linear combination of the two fundamental solutions:

$$\psi(x) = A\psi_e(x) + B\psi_o(x)$$

We express

$$\psi_e \to C_e \operatorname{Cos}(k|x| + \phi_e); \; \psi_o \to \pm C_o \operatorname{Cos}(k|x| + \phi_o)$$

where,

$$\phi_e = \arg \frac{\Gamma\left(i\frac{k}{\alpha}\right)\exp\left(-i\frac{k}{\alpha}\log 2\right)}{\Gamma\left(\frac{\lambda}{2}+i\frac{k}{2\alpha}\right)\Gamma\left(\frac{1-\lambda}{2}+i\frac{k}{2\alpha}\right)} \tag{2.11}$$

and

$$\phi_o = \arg \frac{\Gamma\left(i\frac{k}{\alpha}\right)\exp\left(-i\frac{k}{\alpha}\log 2\right)}{\Gamma\left(\frac{\lambda+1}{2}+i\frac{k}{2\alpha}\right)\Gamma\left(1-\frac{\lambda}{2}+i\frac{k}{2\alpha}\right)} \tag{2.12}$$

From equation (2.3), we have

$$E = \frac{\hbar^2 K^2}{2\mu}$$

and for bound state E < 0. (i.e. Eigenvalues exist for negative energies, we put k = iχ

$$\text{i.e. } E = \frac{-\hbar\chi^2}{2\mu} \tag{2.13}$$

and the parameters in equation (2.8) become real, viz:

$$a = \frac{1}{2}\left(\lambda - \frac{\chi}{\alpha}\right); b = \frac{1}{2}\left(\lambda + \frac{\chi}{\alpha}\right) \tag{2.14}$$

We may again use the asymptotic formulae (2.10a) and (2.10b) in which, however the first term now behaves as exp (α|x|) and the second as exp (-α|x|).

A normalizable solution is therefore possible for χ > 0, if and only If, the factor of the first term vanishes. Since the Γ functions are now all taken for real arguments where poles exist at negative integers, - n (n = 0, 1, 2, 3...) The eigenvalues follow from:

$$\frac{1-\lambda}{2} + \frac{\chi}{2\alpha} = -n \text{ or } \frac{\chi}{\alpha} = \lambda - 1 - 2n \tag{2.15}$$

for even eigenstates.
and

$$1-\frac{\lambda}{2}+\frac{\chi}{2\alpha}=-n \text{ or } \frac{\chi}{\alpha}=\lambda-2-2n \qquad 2.16$$

for odd eigenstates
Hence, the eigenstates (energy terms) become, with a slight change in notations (for even and odd eigenstates)

$$E_n=-\frac{{}^{2}\alpha^{2}}{2\mu}(\lambda-1-n)^{2},\ n=0,1,2,3 \qquad 2.17$$

3.0 ENERGY LEVELS OF A POTENTIAL WELL (BOHR-SOMERFELD QUANTIZATION RULE)

In the semi-classical approximation the determination of discrete energy levels in the potential well V = V(x) reduces to finding the conditions under which real exponential W.K.B.J. solution vanish asymptotically in regions I and III [2]. Thus in region I, we have,

$$\Psi_1(x)=\alpha_1[r(x)]^{-\frac{1}{2}}\exp\left(-\int_x^a r(t)\,dt\right),\quad x<a \qquad (3.1)$$

Implies that

$$\Psi_2(x)=\alpha_2.2[k(x)]^{-\frac{1}{2}}\mathrm{Cos}\left(\int_a^x k(t)\,dt-\frac{\pi}{4}\right),\quad a<x<b \qquad (3.2)$$

And in region III, if,

$$\Psi_3(x)=\alpha_3\,2[r(x)]^{-\frac{1}{2}}\exp\left(-\int_a^x r(t)\,dt\right)x<b \qquad (3.3)$$

Then

$$\Psi_4(x)=\alpha_4.2[k(x)]^{-\frac{1}{2}}\mathrm{Cos}\left(\int_x^b k(t)\,dt-\frac{\pi}{4}\right),\quad a<x<b \qquad (3.4)$$

where

σ_i (I = 1, 2, 3, 4) are non-zero arbitrary constants

where k(x) is defined as:

$$k(x)=\left\{\frac{2\mu}{\hbar^2}[E-v(x)]\right\}^{\frac{1}{2}},\ \text{If } E>v(x)$$

As earlier been stated W.K.B.J. method is a semi-classical approximation, since it is expected to be most useful in the nearly classical limit of large quantum numbers. Equation (3.13) can be used to determine the discrete eigenvalues of any given potential.

4. EIGENVALUES OF A MODIFIED PÖSCHL-TELLER POTENTIAL HOLE USING SEMI-CLASSICAL APPROXIMATION

The energy levels of this potential in semi-classical approximation are obtained from condition (3.13). Where V(x) is as given in equation (2.1); a and b are now the solution of the equation (4.1)

To evaluate the integral

$$J(E)=\int_a^b\left[\frac{2\mu}{\hbar^2}\left\{E+\frac{\hbar^2\alpha^2\lambda(\lambda-1)}{2\mu\cosh^2\alpha x}\right\}\right]^{\frac{1}{2}}dx \qquad (4.2)$$

We should note that for bound state En < 0 for the case under consideration of a discrete spectrum.

On differentiating equation (4.2) with respect to E, the upper and the lower limits of the integral as far as the differentiation is concerned that which depends on E vanishes since the expression under the square root is equal to zero at the turning points a and b. We get thus:

$$J^1(E)=\frac{\mu}{\hbar^2}\int_a^b\left\{\frac{2\mu}{\hbar^2}\left[E+\frac{\hbar^2\alpha^2\lambda(\lambda-1)}{2\mu\cosh^2\alpha x}\right]\right\}^{-\frac{1}{2}}dx \qquad (4.3)$$

where a & b are solutions of

$$E+\frac{\hbar\alpha^2\lambda(\lambda-1)}{2\mu\,\mathrm{Cosh}^2\alpha x}=0 \qquad (4.4)$$

Putting

$$\eta=\mathrm{Sinh}\alpha x \qquad (4.5)$$

Equation (4.3) becomes

$$J'(E)=\frac{\mu}{\alpha\hbar}\int_{\eta_1}^{\eta_2}\frac{d\eta}{\left[2\mu E(1-\eta^2)+\alpha_0\right]^{\frac{1}{2}}} \qquad (4.6)$$

where $\alpha_o = \dfrac{\hbar^2 \alpha^2 \lambda(\lambda - 1)}{2\mu}$ (4.7)

$$J'(E) = \frac{\mu}{\alpha\hbar\left[2\mu(E+\alpha_o)\right]^{1/2}} \int_{\eta_1}^{\eta_2} \frac{d\eta}{\left[1 + \frac{E}{E+\alpha_o}\cdot\eta^2\right]^{\frac{1}{2}}} \tag{4.8}$$

Where η_2 and η_1 are the solutions of (as from equation 4.4)

$$E + \frac{\alpha_o}{1+\eta^2} = 0 \tag{4.9}$$

Now, putting $\eta = [(E + \alpha_o)/E]^{1/2} \sin h\, \theta$ (4.10)
then, equation (4.8) becomes,

$$J'(E) = \frac{\mu}{\alpha\hbar[2\mu E]^{\frac{1}{2}}}[\theta_2 - \theta_1] \tag{4.11}$$

where θ_1 and θ_2 are the solutions of:

$$E + \left[\frac{\alpha_o}{1 + \frac{E+\alpha_o}{E}\operatorname{Sin} h^2\theta}\right] = 0 \tag{4.12}$$

i.e. $E(E + \sigma_o) \sin h^2\, \theta = -(E^2 + E\sigma_o)$ (4.13)

$$\theta_1 = -i\frac{\pi}{2}; \theta_2 = i\frac{\pi}{2} \tag{4.14}$$

Therefore, $J'(E) = \dfrac{\mu\pi}{\alpha\hbar(-2\mu E)^{1/2}}$ (4.15)

$$J(E) = -\frac{\pi}{\alpha\hbar}(-2\mu E)^{1/2} + c_1 \tag{4.16}$$

The constant of integration, C_1 can be determined by observing that for $E = -\alpha_o$, the range of integration in equation (4.2) reduces to the point $x = 0$.
i.e. a and b are solutions of

$$= i\left\{\frac{2\mu}{\hbar^2}\left[v(x) - E\right]\right\}^{\frac{1}{2}} = ir(x), \text{ If } E < v(x) \tag{3.5}$$

As in Figure II, there are supposed to be just two turning points of the classical motion such that,

$\psi_2(x) = \psi_4(x)$, in the interval $a < x < b$.

$$\text{i.e. } \alpha_2 \cos\left(\int_a^x k(t)\,dt - \frac{\pi}{4}\right) = \alpha_4 \,\text{Cost}\left(\int_x^b k(t)\,dt - \frac{\pi}{4}\right) \tag{3.6}$$

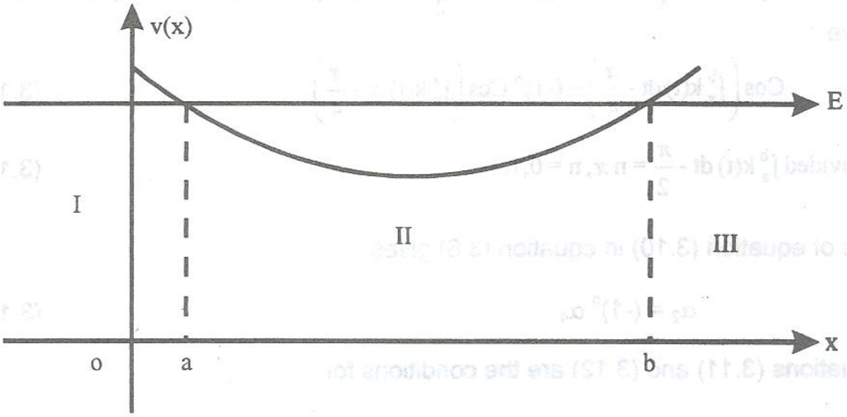


Fig. II: Application of the W.K.B.J. Method to a potential trough: Linear turning points occur at a and b.

Since $[k(x)]^{-\frac{1}{2}} \neq 0$,

The fact that $\int_x^b k(t)\,dt = \int_x^a k(t)\,dt + \int_a^b k(t)\,dt$

$$= -\int_a^x k(t)\,dt + \int_a^b k(t)\,dt \tag{3.7}$$

And $\text{Cos}(-\theta) = \text{Cos}\,\theta$ (is an even function of θ)
Then,

$$\mathrm{Cos}\left(\int_x^b k(t)\,dt - \frac{\pi}{4}\right) = \cos\left[-\int_a^x k(t)\,dt + \int_a^b k(t)\,dt - \frac{\pi}{4}\right] \quad (3.8)$$

$$= Cos\left[\left(\int_a^x k(t)\,dt - \frac{\pi}{4}\right) - \left(\int_a^b k(t)\,dt - \frac{\pi}{2}\right)\right] \quad (3.9)$$

Since Cos $(-\theta)$ = Cosθ

and also from Trigonometrical property,

Cos (A -B) = CosA CosB + SinA SinB

= $(-1)^n$ CosA, Provide B = $n\pi$; n = 0, 1, 2, ...

with $A = \int_a^x k(t)\,dt - \frac{\pi}{4}$, and $B = \int_a^b k(t)\,dt - \frac{\pi}{2}$. In the right hand side of equation (3.9) we have:

$$\mathrm{Cos}\left(\int_x^b k(t)\,dt - \frac{\pi}{4}\right) = (-1)^n\,\mathrm{Cos}\left(\int_a^x k(t)\,dt - \frac{\pi}{4}\right) \quad (3.10)$$

$$\text{Provided } \int_a^b k(t)\,dt - \frac{\pi}{2} = n\pi, n = 0,1,2,... \quad (3.11)$$

use of equation (3.10) in equation (3.6) gives

$$\alpha_2 = (-1)^n \alpha_4 \quad (3.12)$$

equations (3.11) and (3.12) are the conditions for

$\psi_2(x) = \psi_4(x)$, in the interval $a < x < b$.

In particular, equation (3.11) can be written as

$$\int_a^b k(t)\,dt = (n + \frac{1}{2})\pi, n = 0,1,2,3...$$

or, by virtue of equation (3.5a) as:

$$\int_a^b \left[\frac{2\mu}{\hbar^2}(E - v(t))\right]^{\frac{1}{2}} dt = \left(n + \frac{1}{2}\right)\pi, n = 0,1,2,3... \quad (3.13)$$

is the required Both-Sommerfeld quantization rule.

$$-\alpha_o + \frac{\alpha_o}{\text{Cosh}^2 \alpha x} = 0$$

This implies that J-α_o) = 0

Then, $C_1 = \frac{\pi(2\mu\alpha_o)^{\frac{1}{2}}}{\alpha\hbar}$

Condition (3.13) now gives

$$(-2\mu\overline{E}_n)^{1/2} = (2\mu\alpha_o)^{1/2} - \alpha\hbar\left(n + \frac{1}{2}\right)$$

Therefore,

$$\overline{E}_n = -\frac{\alpha^2\hbar^2}{2\mu}\left[\left(\frac{2\mu\alpha_o}{\alpha^2\hbar^2}\right)^{\frac{1}{2}} - \left(n + \frac{1}{2}\right)\right]^2, n = 0,1,2,3,... \quad (4.18)$$

The discrete energy levels for a modified Pöschl-Teller potential hole is,

$$\overline{E}_n = -\frac{\alpha^2\hbar^2}{2\mu}\left\{[\lambda(\lambda -)]^{\frac{1}{2}} - \left(n + \frac{1}{2}\right)\right\}^2, n = 0,1,2,3,... \quad (4.19)$$

The exact eigenvalues E_n of the modified Pöschl-Teller potential hole is as obtained in equation (2.17) and that using semi-classical approximation $\overline{E}_n$ is as obtained in equation (4.19).

If we let $\hbar^2\alpha^2 = 2\mu$ in the two cases, then, we have

$$E_n = -(\lambda - 1 - n)^2, n = 0, 1, 2, \quad (4.20)$$

and

$$\overline{E}_n = \left\{[\lambda(\lambda - 1)]^{\frac{1}{2}} - n - \frac{1}{2}\right\}^2, n = 0,1,2,.. \quad (4.21)$$

We obtain the values of E_n and $\overline{E}_n$ for integer and non-integer λ (odd and even also), where $|\in| = |E_n - \overline{E}_n|$

The discrete energy levels for a modified Pöschl-Teller potential hole is

Table 1: The Values of E_n $\overline{E}_{n'}$ and $|\epsilon|$ with different values of λ.

n	$\lambda = 1.5$			$\lambda = 2$			$\lambda = 5$			$\lambda = 10$		
	E_n	$\overline{E}_{n'}$	$\|\epsilon\|$	E_n	$\overline{E}_{n'}$	$\|\epsilon\|$	E_n	$\overline{E}_{n'}$	$\|\epsilon\|$	E_n	$\overline{E}_{n'}$	$\|\epsilon\|$
0	- 0.250	- 0.137	0.113	-1.000	-0.836	0.164	-16.000	-15.778	0.222	-81.000	-80.763	0.237
1	- 0.250	- 0.397	0.147	0.000	-0.007	0.007	-9.000	-8.834	0.166	-64.000	-63.790	0.210
2	- 2.250	- 2.657	0.407	-1.000	-1.179	0.179	-4.00	-3.889	0.111	-49.000	-48.816	0.184
3	- 6.250	- 6.917	0.667	-4.000	-4.351	0.351	-1.000	-0.945	0.055	-36.000	-35.842	0.158
4	-12.250	- 13.177	0.927	-9.000	-9.522	0.522	0.000	-0.001	0.001	-25.000	-24.869	0.131
5	-20.250	- 21.437	1.187	-16.000	-16.694	0.694	-1.000	-1.057	0.057	-16.000	-15.895	0.105
6	-30.250	- 31.697	1.447	-25.000	-25.865	0.865	-4.000	-4.112	0.112	-9.000	-8.921	0.079
7	-42.250	- 43.957	1.707	-36.000	-37.037	1.037	-9.000	-9.168	0.168	-4.000	-3.946	0.054
8	-56.250	- 58.217	1.967	-49.000	-50.208	1.208	-16.000	-16.224	0.224	-1.000	-0.974	0.026
9	-72.250	- 74.447	2.227	-64.000	-65.380	1.380	-25.000	-25.279	0.279	0.000	0.000	0.000
10	-90.250	- 92.737	2.487	-81.000	-82.552	1.552	-36.000	-36.335	0.335	-1.000	-1.027	0.027

5. DISCUSSION OF RESULTS.

The even and odd eigenstates as in equations (2.15) and (2.16) can be combined as one as in equation (2.17) i.e. $E_n = -\frac{\hbar^2\alpha^2}{2\mu}(\lambda - 1 - n)^2, n = 0,,2,...$ which is identical with equation (4.19), Hence the semi-classical approximation method gives nearly the exact energy levels for the modified Pöschl-Teller potential hole.

Table I shows that the eigenvalues are less than zero. For the exact results (E_n), it is noted that for integer λ there is always one eigenvalue ($n = \lambda - 1$) lying at zero energy while on the other hand, semi-classical approximation method nearly (asymptotically) lying at zero energy.

$|\in|$ is the modulus of the difference between the corresponding values of n for E_n and $\overline{E_n}$.

For non-integer λ (λ = 1:5), $|\in|$increases as n increases, likewise λ = 2, λ = 5 but for λ = 10 the maximum value of $|\in|$= 0.237 occurs when n = 0 and decreases with increasing n to $|\in|$ = 0.027 (when n = 10), This indirectly confirms the description of the asymptotic approximation method as a semi-classical (quasi-classical) approximation, where results are most accurate in the nearly classical limit of large quantum numbers; These eigenvalues directly depend on the value of λ the most accurate results are obtained for large value of λ where λ is an integer or not (even or odd too).

6.0 CONCLUSION

The semi-classical (path-integral) approximation is the relatively simple Bohr-Sommerfeld condition for the energy spectrum in a quantitative manner in which the quantization procedure can be performed with these W.K.B.J. formulae (solutions). To get an estimate of the quality of the W.K.B.J approach to obtain the discrete energy values, it is interesting to compare the results with the exact results. There are no essential or remarkable differences between the exact eigenvalues and W.K.B.J. eigenvalues for modified Pöschl-Teller potential hole as n increases as well as λ The method (W.K.B.J) presented here is to serve as alternative that depends quantum mechanically on n and λ.

The accuracy of W.K.B.J approximation method is best assessed by comparing the W.K.B.J discrete energy values of this potential with those obtained either by solving the Schrödinger equation numerically, or whenever possible, using exact analytic expressions and since exact solution using this potential is possible we then used the exact analytic expressions to compare.

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REFERENCES

[1] Giler, S. : "Generalised W.K.B.J. Formulae"., J. Phys. A: math. Gen 21 (1988) Pgs. 909 - 930.

[2] Merzbacher, E. : "Quantum Mechanics". 2nd edition. (John Wiley and sons, Inc., New York, 1970) Chapter 7.

[3] Cocolicchio, D. & Viggiano, M. : "WKB approximation without divergences." International Journal of Theoretical Physics, Vol. 36,No. 12 (1997) Pgs 3051 - 3063.

[4] Giler, S., Kosinski, P., Rembielinski, J. & Manlanka, P.: "The tiny effect of the ground-state energy in the case of a quantum Mechanical Potential with broken supersymmetry", Lodz University Preprint IF UL - 2/86, 1986.

[5] Farina, J.E.G: "Transmission probability and traversal time in Scattering by a 1 - d potential of finite range"., J. Phys A: math. Gen 21 (1988), Pgs. 2547 - 2558.

[6] Méndez, B. & Dominguez-Adame, F.: "Numerical Study of electron tunnelling through heterostructures"., Am. J. Phys. Vol. 62 No. 2 (1994) Pgs. 143 - 147.

[7] Oyewumi, K. J. & Bangudu, E. A.: "Application of Asymptotic approximation method to harmonic and anharmonic (quartic) Oscillators". To appear 1998.

[8] Ghatak, A.K., Sauter, E.G., & Goyal, I.C.: "Validity of the JWKB formula for a triangular potential barrier". Eur. J. Phys. 18 (1997) Pgs 199 - 204.

[9] Birx, D.L. & Houk, T.W.: "The eigenvalues of the Upper and Lower bounds of anharmonic oscillator potential". Am. J. Phys. 45, (1977) Pg. 1070.

[10] Robnik, M. & Salasnich, L.: "WKB to all orders and the accuracy of the semi classical quantization." J. Phys. A: Math. Gen 30 (1997) Pgs 1711 - 1718.

[11] Chebotarev, L.V.: "Total delay time and tunnelling time for non-rectangular potential barriers," J. Phys. A: Math. Gen 29 (1996) Pgs 1465 - 1486.

[12] Grypeos, M.E. & Liolios, Th. E.: "Application of the Quantum Mechanical Hypervirial Theorems to Even-power series potentials" International Journal of Theoretical Physics, Vol. 36, No. 10 (1997) Pgs 2051 - 2066.

[13] Lalazissis, G.A., Grypeos, M.E., & Massen, S.E.: Physical Review C, 37, (1988) Pg. 2098.

[14] Lalazissis, G.A.: Physical Review C, 49 (1994) Pg. 1412.

[15] Lalazissis, G.A.: Physical Review C. 48 (1993) Pg. 198.

[16] Pöschl, G., Teller, E.: Z. Physik Vol. 83 No. 143, (1933) Pg. 95.

[17] Flügge, S. : "Practical Quantum mechanics". 2nd Edition, Springer-Verlag (1974), Pgs. 94 - 100.

[18] Abramowitz, M. & Stegun, I.A.: "Handbook of Mathematical Functions with Formulas, Graphs & Mathematical Tables". National Bureau of Stds; Applied mathematics Series 55. (1970) Chapter 15.

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COMPUTATION OF THE KERNEL FUNCTION BY INVERSION OF RESISTIVITY FIELD DATA FROM SCHLUMBERGER CONFIGURATION

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ABSTRACT

A major problem in geophysical investigations using electrical methods is the interpretation of results in terms of lithological variation with depth. Interpretation is commonly found by matching practical field curves with theoretically generated curves from resistivity kernel functions. The kernel function cannot be measured in the field but has to be obtained from a transformation of measured apparent resistivities. Several kernel functions have been identified. The need to formulate an optimum kernel function leads to continued research on this subject. This work reviews major previous kernel functions in literature with their peculiar problems . Apparent resistivity value is then theoretically derived as an expression involving an integral over the kernel function, which contains all needed information about the present configuration of the earth. The integral equation is solved to get the kernel function. Theoretical curves generated with the kernel function were matched with practical curves from field data obtained in Owan West Local Government of Edo State, Nigeria. The interpretation based on this kernel function agreed very satisfactorily with well logs in this area of Edo State.

INTRODUCTION

The kernel function is a function of only theoretical resistivities and theoretical thicknesses of earth layers for the purpose of drawing theoretical apparent resistivity curves. These curves are then matched with curves obtained from actual field measurements for the purpose of interpretation.

A differential equation which is the basis of all resistivity-prospecting with direct current is given by

$$\nabla.(\sigma_{ij}\nabla V) = 0 \tag{1}$$

where V is potential and σ is conductivity variation with depth. In the isotropic case where the conductivity at a point on the ground is independent of direction, equation (1) reduces to Laplace equation

$$\nabla^2 V = 0 \tag{2}$$

The solution of equations (1) and (2) may be developed for a particular model of the earth.

Slichter (1933) and Langer (1933) derived the solution of equation (1) as

$$V(r) = \frac{I\rho_i}{2\pi_0}\int_0^{\infty} K(\lambda)J_0(\lambda r)d\lambda \qquad (3)$$

where $K(\lambda)$ is the Slichter kernel, J_0 is the zero-order Bessel function of the first kind, r is distance and $\lambda=(\rho_t/\rho_l)^{1/2} = (\sigma_l/\sigma_t)^{1/2}$ = anisotropy, ρ is resistivity while σ = conductivity. The subscript ι means parallel to the layers while the subscript t means tangential to the layers.

Pekeris (1940) adopted a graphical method based on Slichter kernel for the analysis of horizontal layers. The method shows that a function of the kernel

$$f(\lambda) = \frac{K(\lambda)+1}{K(\lambda)-1} \qquad (4)$$

should be plotted against λ on log-linear graph paper. The resolution of such a graph was just fairly satisfactory.

Some other workers on kernel function include Stevenson (1934), Keck and Colby (1942) and Vozoff (1958). The deficiencies in these previous works, according to Koefoed (1965) were

(a) the kernel had always been defined in terms of observed potentials, and

(b) the kernel had been computed by numerical integration.

Koefoed (1965a) removed these two deficiencies by applying Hankel inversion to the resistivity formula of Stefanesco *et al.* (1930) to define kernel function in terms of observed apparent resistivities for two-layered earth. Even then he could not compute two-layer apparent resistivity curves. As such, he used approximation method to a field curve by a sum of irrational or exponential functions which he called partial apparent resistivity functions, $\Delta\rho_a(r)$ and obtained

$$\rho_a(r) = \rho_1 + \Sigma\Delta\rho_a(r) \qquad (5)$$

where ρ_1 is the first layer resistivity. The results of Koefoed (1965a) were only partially satisfactory. As such, Koefoed (1965b, 1968) subsequently abandoned curve-matching in the kernel domain in favour of the graphical method of Pekeris (1940).

A universally acceptable kernel function is yet to be identified, not withstanding, the common medium of interpreting resistivity field data to date is by matching field curves with one type of kernel function curve or the other.

A theoretical solution for a kernel function of a two layer earth is presented in this work. Theoretical curves thus generated were matched with field curves and the results compared very satisfactorily with well log data.

THEORETICAL ANALYSIS

Apparent resistivity can be written in terms of Bessel function (Bowman (1958)) as

$$\rho_a(r) = \rho_1 \left\{ 1 + 4r \int_0^{\infty} [\theta_n(\lambda)^{-1/2} \theta_n(1/2\lambda)] J_0(\lambda r) d\lambda \right\} \qquad (6)$$

where $J_0(\lambda r)$ = the Bessel function of order 0 and first kind, $\theta_n(\lambda)$ = the kernel function (n = number of layers)

Fig. 1 shows the Schlumberger configuration on which the computation of the kernel function by inversion of resistivity field data is based.

The kernel function is defined as a ratio of polynomials,

$$\theta_n(\lambda) = \frac{P_n(\lambda)}{H_n(\lambda)} \qquad (7)$$

and recursively with $u = e^{-2\lambda}$, we have

$H_1(u) = 1.$

$P_1(u) = 0$

$H_m(u) = H_{m-1}(u) - k_{m-1} u^{dm-1} H_{m-1}(u^{-1})$

$P_m(u) = P_{m-1}(u) + k_{m-1} u^{dm-1} (P_{m-1}(u^{-1}) + H_{m-1}(u^{-1}))$ $\quad m = 2 \ldots n$

Explicitly for n = 2 we have

$$\theta_2(\lambda) = \frac{k_1 e^{-2d_1\lambda}}{1 - k_1 e^{-2d_1\lambda}} \qquad (8)$$

and for n = 3 we have

$$\theta_3(\lambda) = \frac{k_1 e^{-2d_1\lambda} + k_2 e^{-2d_2\lambda}}{1 - k_1 e^{-2d_1\lambda} - k_2 e^{-2d_2\lambda} + k_1 k_2 e^{-2(d_2-d_1)\lambda}} \qquad (9)$$

where k = reflection factor, and d = depth, (see fig.1)

In seeking a solution for the kernel function we assume that the function ρ_a (r) is given and we obtain an expression for the kernel function $\theta_n(\lambda)$ of equation (6).

We introduce the functions

$$f(\lambda) = \theta_n(\lambda) - \tfrac{1}{2}\,\theta_n(\tfrac{1}{2}\lambda) \tag{10}$$

and

$$c(\lambda) = \rho_1(4f(\lambda) + 1) \tag{11}$$

Then, from equation (6) we have

$$\rho_a(r) = \int_0^\infty c(\lambda) J_0(\lambda r)\, d\lambda \tag{12}$$

The Fourier-Bessel integral states (Bowman (1958)).

$$\int_0^\infty \left\{ \int_0^\infty F(x) J_0(xt) \right\} (xt) x\, dx\, J_0(st)\, t\, dt = F(s) \tag{13}$$

Multiplying equation (12) by $J_0(\lambda r)$ and integrating over r we have

$$\int_0^\infty \rho_a(r) J_0(\lambda r)\, dr = \int_0^\infty r \left\{ \int_0^\infty c(t) J_0(tr)\, dt \right\} J_0(\lambda r)\, dr$$

$$= \int_0^\infty \left\{ \int_0^\infty \frac{c(t)}{t} J_0(tr)\, t\, dt \right\} J_0(\lambda r)\, r\, dr$$

$$= \frac{c(\lambda)}{\lambda} \tag{14}$$

Thus,

$$c(\lambda) = \lambda \int_0^\infty \rho_a(r) J_0(\lambda r)\, dr = \int_0^\infty \rho_a\left(\frac{t}{\lambda}\right) J_0(t)\, dt \tag{15}$$

From equation (11) therefore, we have

$$f(\lambda) = \frac{c(\lambda) - \rho_1}{4\rho_1} \tag{16}$$

Thus, equations (15) and (16) can now be used for the calculation of $f(\lambda)$.
From equation (10) we can easily derive the equation

$$\theta_n(\lambda) \quad = \quad \sum_{k=0}^{\infty} \frac{1}{2^k} f \frac{1}{2^k} \tag{17}$$

$$\therefore\ \theta_n(\lambda) = f(\lambda) + \tfrac{1}{2}\theta_n\left(\tfrac{\lambda}{2}\right) \tag{18}$$

$$\theta_n\left(\tfrac{\lambda}{2}\right) = f\left(\tfrac{\lambda}{2}\right) + \tfrac{1}{2}\theta_n\left(\tfrac{\lambda}{4}\right)$$

$$\vdots$$

Thus

$$\theta_n(\lambda) = f(\lambda) + \tfrac{1}{2}\left(f\left(\tfrac{\lambda}{2}\right) + \tfrac{1}{2}\theta_n\left(\tfrac{\lambda}{4}\right)\right)$$

$$= f(\lambda) + \tfrac{1}{2} f\left(\tfrac{\lambda}{2}\right) + \tfrac{1}{4}\theta_n\left(\tfrac{\lambda}{4}\right) \tag{19}$$

and so on. The series in equation (17) is convergent since $f(\lambda) \to$ constant when $\lambda \to 0$.

The value for $c(\lambda)$ is given by equation (15). However, for a two-layer earth we have

$$c(\lambda) \quad = \quad \int_0^{\infty}\left(\rho_a\left(\tfrac{t}{\lambda}\right) - \rho_{end}\left(\tfrac{t}{\lambda}\right)\right)J_0(t)\,dt + \int_0^{\infty}\rho_{end}\left(\tfrac{t}{\lambda}\right)J_0(t)\,dt \tag{20}$$

where $\rho_{end}(r)$ is the apparent resistivity curve for a two-layer earth. The subscript "end" means at the end.

$$c_{end}(\lambda) = \int_0^{\infty}\rho_{end}\left(\tfrac{t}{\lambda}\right)J_0(t)\,dt = \rho_{1end}\left(4\left(\theta_{end}(\lambda) - \tfrac{1}{2}\theta_{end}\left(\tfrac{\lambda}{2}\right)\right) + 1\right) \tag{21}$$

where

$$\theta_{end}(\lambda) = \frac{k_{end}\, e^{-2d_{end}\lambda}}{1 - k_{end}\, e^{-2d_{end}\lambda}} \tag{22}$$

From equation (20) we write

$$c(\lambda) = \int_0^{\infty}\left(\rho_a\left(\tfrac{t}{\lambda}\right) - \rho_{end}\left(\tfrac{t}{\lambda}\right)\right)J_0(t)\,dt + c_{end}(\lambda) \tag{23}$$

We note that $\rho_a(t) - \rho_{end}(t) = 0$ for $t \geq r_{end}$
For the integral in equation (23) we put

$$D(\lambda) = \int_0^{\infty}\left(\rho_a\left(\tfrac{t}{\lambda}\right) - \rho_{end}\left(\tfrac{t}{\lambda}\right)\right)J_0(t)\,dt$$
$$= \sum_{j=0}^{\infty} D_j(\lambda) \qquad (24)$$

where

$$D_j(\lambda) = \int_{z_j}^{z_{j+1}}\left(\rho_a\left(\tfrac{t}{\lambda}\right) - \rho_{end}\left(\tfrac{t}{\lambda}\right)\right)J_0(t)\,dt. \qquad (25)$$

(z_j = the j^{th} zero order of $J_0(t)$; $z_0 = 0$).

The series in equation (24) will eventually form an alternating series with decreasing terms. The Euler transformation is then used to speed up the convergence.

The problem with equation (17) is that f(0) can be large.

In equation (23) $c(\lambda) \equiv c_{end}(\lambda)$ for small values of λ, therefore, we assume

$$c(\lambda) = c_{end}(\lambda) \text{ for } \lambda \leq \lambda_1 \qquad (26)$$

Calculating $\theta_n(\lambda)$ for $\lambda \leq \lambda_1$

To calculate $\theta_n(\lambda)$ for $\lambda \leq \lambda_1$ we have, from equations (16), (17) and (26)

$$\theta_n(\lambda) = \sum_{m=0}^{\infty} \frac{1}{2^m} f\left(\frac{\lambda}{2^m}\right) = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{c\left(\frac{\lambda}{2^m}\right) - \rho_1}{4\rho_1}$$

$$= \sum_{m=0}^{\infty} \frac{\dot{\lambda}}{2^m} \frac{c_{end}\left(\frac{\lambda}{2m}\right) - \rho_1}{4\rho_1} \qquad \text{for } \lambda \leq \lambda_1$$

Using equation (21) and noting that

$$\tfrac{1}{2}\sum_{m=0}^{\infty} \frac{1}{2^m} \frac{k_{end} e^{-d_{end}\frac{\lambda}{2m}}}{1 - k_{end} e^{-d_{end}\frac{\lambda}{2m}}} = \sum_{m=0}^{\infty} \frac{k_{end} e^{-2d_{end}\frac{\lambda}{2m}}}{1 - k_{end} e^{-2d_{end}\frac{\lambda}{2m}}}$$

we have

$$\theta_n(\lambda) = \frac{\rho_{1end} - \rho_1}{2\rho_1} + \left(\frac{\rho_{1end}}{\rho_1}\right)\frac{k_{end}e^{-2d_{end}\lambda}}{1-k_{end}e^{-2d_{end}\lambda}} \qquad (27)$$

Calculating $\theta_n(\lambda)$ for $\lambda \to \infty$

When $\lambda \to \infty$ then $f(\lambda) \to 0$. We determine a value $\lambda = \lambda_0$ such that $|f(\lambda)| \le \epsilon_1$ for $\lambda \ge \lambda_0$

We then calculate $f\left(\frac{\lambda_0}{2^{k-1}}\right)$ $k = 1, 2, ---$

until $\frac{\lambda_0}{2^{k_0-1}} \le \lambda_0$

Let that value of k be $k = k_0$, then, according to equation (27) we obtain the value of $\theta_n\left(\frac{\lambda_0}{2^{k_0-1}}\right)$ and from equation (17) we have

$$\theta_n\left(\frac{\lambda_0}{2^{k-1}}\right) = f\left(\frac{\lambda_0}{2^{k-1}}\right) + \tfrac{1}{2}\theta_n\left(\frac{\lambda_0}{2^k}\right), \quad k = k_0-1, --- 1 \qquad (28)$$

Thus, theoretical apparent resistivity curves can be computed with equation (27) for $\lambda \le \lambda_1$ and with equation (28) for $\lambda \to \infty$.

EXPERIMENTAL WORK

Fig. I, gives the Schlumberger electrode array. Osemeikhian and Asokhia (1994) derived the apparent resistivity values for field data as

$$\rho_a = \{\pi(CD)V/I\}\{(L/CD)^2 - 0.25\} \qquad (29)$$

where L is half the current electrode separation, CD is the potential electrode separation, V is potential and I is the current generated into the earth.

Data were collected at Ozalla-Ora, Uhonmora and Sabongidda-Ora, all in Owan-West Local Government areas of Edo State.

Equations (27) and (28) were incorporated into a computer programme in the basic language for generating theoretical kernel curves that were matched automatically with field curves drawn with the computer using equation (29).

RESULTS AND DISCUSSION

A plot of apparent resistivity curves against electrode spacing shows that the data curves for Ozalla-Ora, Uhonmora and Sabongidda - Ora were of the H-type, Q-type and H-type respectively.

The shapes of these curves serve as guide in formulating model parameters for the theoretical curves using the kernal function.

Table 1. Summarises the model parameters and the results.

Hand-dug wells in these three areas confirm that the aquifers are within the range of the results obtained.

Full results and field curves for all stations are given in table 2 to 4 and in figs. 2 to 4 respectively.

TABLE 1 Summary of Results

Station	Ozalla-Ora	Sabongida-Ora	Uhonmora
1st layer model depth	1m	1m	1m
2nd layer model depth	5m	12m	8m
1st layer model ρ_a	567Ω-m	221Ω-m	311Ω-m
2nd layer model ρ_a	272Ω-m	50Ω-m	160Ω-m
3rd layer model ρ_a	1466Ω-m	82Ω-m	10Ω-m
Approximate depth of aquifer	6m	13m	9m
Percentage error between actual curve and theoretical curve	5.85	5.65	6.44

CONCLUSION

The kernel function presented in this paper has a good fit with field data. The percentage errors between theoretical curves and field data curves were less than 7% in all the stations. It is recommended that research work should continue until a much smaller percentage error between field curves and data curves is realizable.

TABLE 2

PROJECT TITLE : VERTICAL ELECTRICAL SOUNDINGS (VES) –SABO
RES VERSUS ELECTRODE SPACING WORKS
MODEL PARAMETERS FOR SCHLUMBERGER SURVEY

LAYER	RESISTIVITY (ohm-m)	THICKNESS(m)
1.	311	1
2.	160	8
3.	10	

AB/2 (meters)	OBSERVED VALUES (ohm-m)	COMPUTED VALUES (ohm-m)
1	221	198.46
1.47	184	169.11
2.15	140	126.74
3.16	100	86.46
4.64	80	62.86
6.81	60	54.53
10	55	53.13
14.7	60	54.32
21.5	68	57.76
31.6	61	63.22
46.4	59	69.15
68.1	64	74.1
100	65	77.56
147	82	79.69

RMS Error 6.45%

Field Measurements By M.B. ASOKHIA, ONWUKA, F.O & UJUANBI, O
Data Interpreted By M.B. ASOKHIA, ONWUKA, F.O. & AZI, S.O.

TABLE 3

PROJECT TITLE: VERTICAL ELECTRICAL SOUNDING (VES) --OZALLLA RES VERSUS ELECTRODE SPACING WORKS MODEL PARAMETERS FOR SCHLUMBERGER SURVEY

LAYER	RESISTIVITY (Ohm-m)	THICKNESS(m)
1.	567	1
2.	272	5
3.	1466	

AB/2(meters)	Observed Values (Ohm-m)	Computed Values (Ohm-m)
1	567	535.48
1.47	428	492.26
2.15	348	428.7
3.16	272	366.96
4.64	302	335.92
6.81	384	351.18
10	481	417.5
14.7	615	529.83
21.5	708	672.25
31.6	867	832.8
46.4	1000	993.89
68.1	1210	1140.25
100	1466	1259.44
147	1522	1345.59

RMS Error 5.98%

Field Measurements By M.B. ASOKHIA, ONWUKA, F.O & UJUANBI, O

Data Interpreted By M.B. ASOKHIA, ONWUKA, F.O. & AZI, S.O.

TABLE 4
PROJECT TITLE: VERTICAL ELECTRICAL SOUNDING (VES) --UHONMORA
RES VERSUS ELECTRODE SPACING WORKS
MODEL PARAMETERS FOR SCHLUMBERGER SURVEY

LAYER	RESISTIVITY (Ohm-m)	THICKNESS (m)
1	221	1
2	50	12
3	82	

AB/2(meters)	Observed Values (Ohm-m)	Computed Values (Ohm-m)
1	311	293.44
1.47	290	271.3
2.15	276	238.05
3.16	263	202.59
4.64	204	175.58
6.81	146	156.08
10	115	132.37
14.7	72	95.38
21.5	59	53.84
31.6	30	24.28
46.4	15	12.72
68.1	11	10.77
100	10	10.63
147	10	10.33

RMS Error 5.65%

Field Measurements by M.B. ASOKHIA, ONWUKA, F.O & UJUANBI, O
Data Interpreted By M.B. ASOKHIA, ONWUKA, F.O. & AZI, S.O.

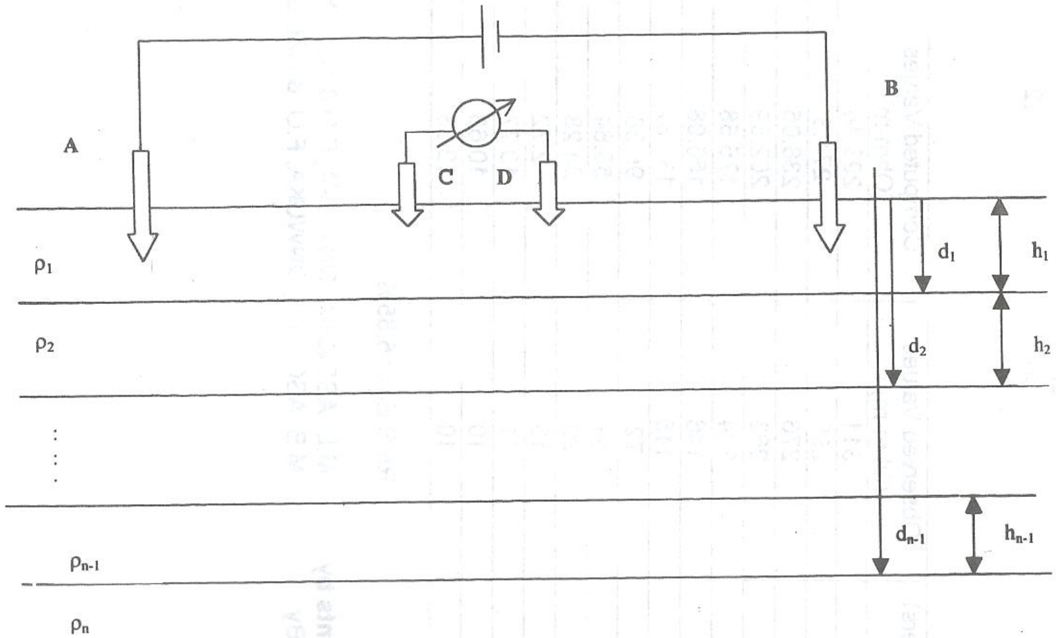

Fig. 1 Schlumberger electrode array pattern

A, B = Current Electrodes; C, D = Potential Electrodes

ρ_i = *the resistivity of the i-th layer*

d_i = *the depth to the interface between the i-th and the (i+1)-st layer*

h_i = *the thickness of the i-th layer*

$$k_i = \frac{\rho_{i+1} - \rho_i}{\rho_{i+1} + \rho_i}$$ = *refection factor, i = 1, ... n-1*

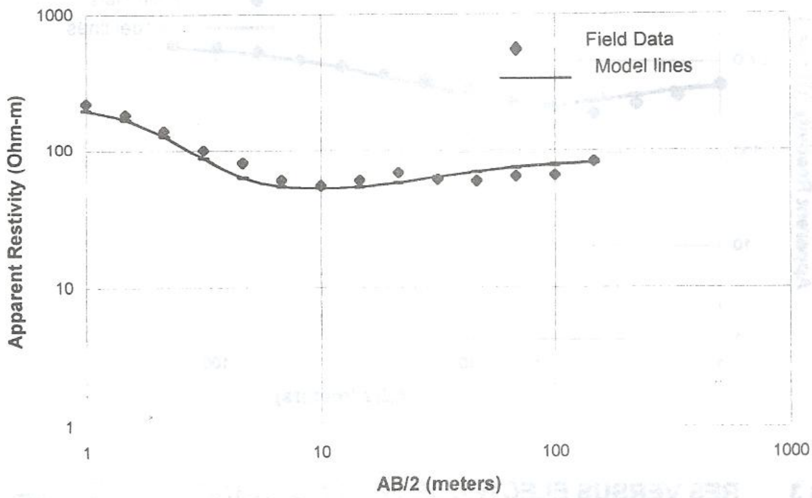

FIG 2 RES VERSUS ELECTRODE SPACING. LOCATION - SABO

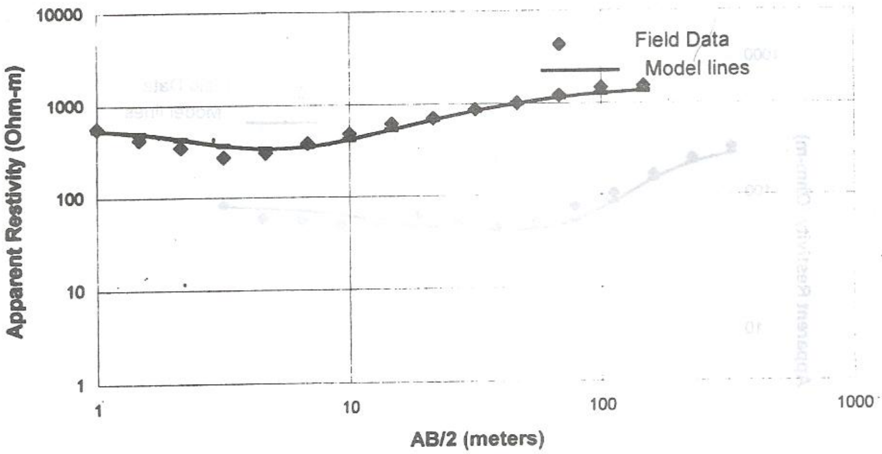

FIG 3 RES VERSUS ELECTRODE SPACING. LOCATION - OZALLA

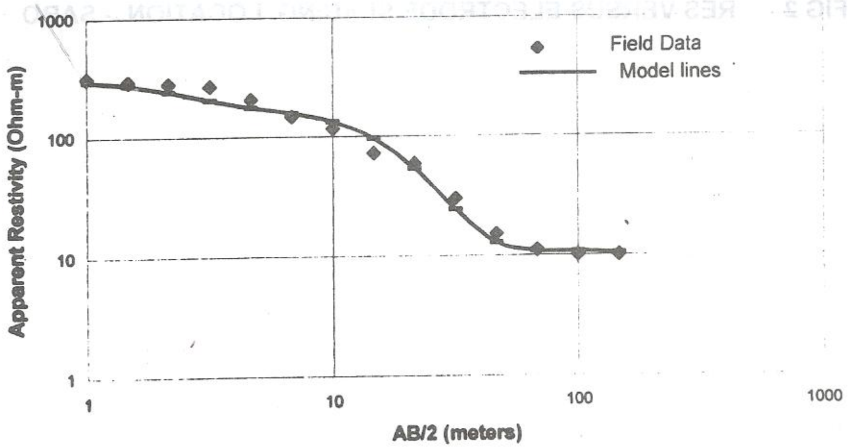

FIG 4 RES VERSUS ELECTRODE SPACING. LOCATION UHONMORA

REFERENCES:

1. BOWMAN, F; (1958): "Introduction to Bessel fucntions" Dover, New York.
2. KECK, W.G. and COLBY, W.F., (1942): "The depth dependence of earth conductivity upon surface potential data." *J. Appl. Phys., Vol. 13,* pp179 – 188.
3. KOEFOED, O. (1965a): "A Semi-direct method of interpreting resistivity observations" Geophys. Prosp. Vol 13. No 2 pp 259 – 282.
4. ------ (1965b): "Direct methods of interpreting resistivity observations". Geophys. Prosp. Vol. 13 No 4. Pp. 568 – 591.
5. ------ (1968): "The application of the kenel function in interpreting geoelectrical resistivity measurements." Gebriider Borntraeger. Berlin. Geoexploration Monographs series 1. No. 2, 108p
6. LANGER, R.E., (1933): "An inverse problem in differential equations" Bull. Math. Soc., Vol. 39, pp 814 - 820.
7. OSEMEIKHIAN, J.E.A. and ASOKHIA, M.B. (1994): "Applied Geophysics" Samtos Services Ltd., Ilupeju, Lagos.
8. PEKERIS, C.L. (1940): "Direct method of interpretation in resistivity prospecting" Geophys., Vol. 5, No. 1, pp 31 –42.
9. SLICHTER., L.B., (1933): "The interpretation of the resistivity prospecting method for horizontal structures." Physics, Vol. 4 pp 307 – 327.
10. STEFANESCO, S.; SCHLUMBERGER, C. and SCHLUMBERGER, M. (1930): "Sur la distribution electrique potentielle autour d'une prise de terre ponctuelle dans un terrain a couches horizontales homogene et isotropes," J. de physique et le Radium, series 7, Vol. 1, pp 132 – 140.
11. STEVENSON A. F., (1934): "On the theoretical determination of earth resistance from surface potential measurements" Physics, Vol. 5, pp. 114 – 124.
12. VOZOFF, K.; (1958): "Numerical resistivity analysis: horizontal layers." Geophysics Vol. 23. No. 3, pp 536 – 556