J. Nig. Ass. Math. Phys.
Vol. 3 (1999)

ON DERIVATION OF NEW NUMERICAL INTEGRATOR FOR SOLVING LINEAR STIFF FIRST ORDER INITIAL VALUE PROBLEM WITH CONSTANT MATRIX OF ORDER 2

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ABSTRACT

In the light of **Fatunla** [1], a new numerical integrator for solving stiff first order initial value problem with constant matrix of order 2 was derived based on the interpolating function:

$$f(x_n) = A\exp(\lambda_1 x_n) + B\exp(\lambda_2 x_n) + C$$

where A,B,C are vectors with real entries and λ_2, is a complex eigen value with λ_2, the complex conjugate of λ_1. Application of this linear integrator to stiff first order initial value problem with constant matrix of order 2 gives an encouraging approximation to the exact solution. The integrator is exponentially fitted and L-stable.

1.0 INTRODUCTION

In[1], the author proposed an integration scheme which effectively copes with linear systems of ordinary differential equations with widely varying eigenvalues. The integration scheme in [1] is based on local representation of the theoretical solution to the initial value problem by an interpolating function.

$$F(x_n) = \left(I - e^{\lambda_1 x_n}\right)A + \left(I - e^{\lambda_2 x_n}\right)B + C$$

As a follow up, in this paper we shall adopt the interpolating function

$$F(x_n) = A\exp(\lambda_1 x_n) + B\exp(\lambda_2 x_n) + C$$

to derive a new integrator for solving linear stiff first order initial value problem with constant matrix of order 2.

2.0 STATEMENT OF THE PROBLEM

Consider the stiff first order initial value problem

$$y' = f(x,y),\ y(a) = \beta \tag{2.1}$$

where

$$a \le x \le b,\ y \in (-\infty, \infty),\quad \beta \in R$$

with

$$f(x,y) = A^* y;\ y = (y_1, y_2)^T \text{ and } \beta = (\beta_1, \beta_2)^T, \tag{2.2}$$

A^* is a constant matrix of order 2.

DEFINITION 2.2 Lambert [3]

The initial value problem (2.1) is said to be stiff over the interval $a \le x \le b$ if for every x in the interval the eigenvalues λ_i, $i = 1,2$ satisfy the following conditions:

$$\text{(i)}\quad \mathrm{Re}\{\lambda_1, \lambda_2\} < 0$$

$$\text{(ii)}\quad \max\left|\frac{\mu_i}{v_i}\right| >> 1 \tag{2.3}$$

with

$$\lambda_1 = \mu_1 + iv_1$$
$$\lambda_2 = \mu_2 - iv_2 \tag{2.4}$$

3.0 DERIVATION OF THE NEW INTEGRATOR

Let y_n be the approximation to the exact solution $y(x_n)$ of (2.1) at the point $x = x_n$; n = 0, 1,2...

Consider the interpolating function

$$F(x_n) = A\exp(\lambda_1 x_n) + B\exp(\lambda_2 x) + C \tag{3.1}$$

Fatunla [1] and Aladeselu [4] suggested the following desirable constraints on (2.1) and (3.1).

$$y_{n+j} = F(x_{n+j}),\quad j = 0,1 \tag{3.2}$$

$$f_n = F'(x_n) \tag{3.3}$$

where

$$f_n = f(x_n, y_n)$$

Now for j = 0,1:

$$y_n = A\exp(\lambda_1 x_n) + B\exp(\lambda_2 x_n) + C \quad (3.4)$$

$$y_{n+1} = A\exp(\lambda_1 x_{n+1}) + B\exp(\lambda_2 x_{n+1}) + C \quad (3.5)$$

and

$$f_n = A\lambda_1 \exp(\lambda_1 x_n) + B\lambda_2 \exp(\lambda_2 x_n) \quad (3.6)$$

$$f_n' = A\lambda_1^2 \exp(\lambda_1 x_n) + B\lambda_2^2 \exp(\lambda_2 x_n) \quad (3.7)$$

Adopting discretization process

$$x_{n+1} = x_n + h \quad (3.8)$$

where n = 0,1,2,...... and $h \in (0,1]$.
We obtain

$$\Delta y_n = A\exp(\lambda_1 x_n)[\exp(\lambda_1 h) - 1] + B\exp(\lambda_2 x_n)[\exp(\lambda_2 h) - 1] \quad (3.9)$$

Since $\lambda_1 \neq \lambda_2$, we obtain from (3.6) and (3.7) the following expressions;

$$A\exp(\lambda_1 x_n) = \frac{1}{N}\left[\lambda_2^2 f_n - \lambda_2 f_n'\right] \quad (3.10)$$

$$B\exp(\lambda_2 x_n) = \frac{1}{N}\left[-\lambda_1^2 f_n - \lambda_1 f_n'\right] \quad (3.11)$$

where

$$N = \lambda_1\lambda_2^2 - \lambda_1^2\lambda_2 \quad (3.12)$$

Substituting expression (3.10) and (3.11) into equation (3.9), we have

$$\Delta y_n = \frac{1}{N}\left[\lambda_2^2 f_n[\exp(\lambda_1 h) - 1] + \lambda_1^2 f_n[1 - \exp(\lambda_2 h)]\right] +$$

$$\lambda_2 f_n'[1 - \exp(\lambda_1 h)] + \lambda_1 f_n'[\exp(\lambda_2 h) - 1] \quad (3.13)$$

$$\Delta y_n = G(h) f_n + H(h) f_n' \quad (3.14)$$

where

$$G(h)=\frac{[\exp(\lambda_1 h)-1]\lambda_2^2-[\exp(\lambda_2 h)-1]\lambda_1^2}{N} \quad (3.15)$$

$$H(h)=\frac{[\exp(\lambda_2 h)-1]\lambda_1-[\exp(\lambda_1 h)-1]\lambda_2}{N} \quad (3.16)$$

Consider the eigenvalues of the constant matrix, A*:

$$\lambda_1 = u + iv \quad \text{and} \quad \lambda_2 = u - iv \quad (3.17)$$

Adopting the eigenvalues in equation (3.15) and 3.16), we have:

$$N = -2v(u^2 + v^2)i \quad (3.18)$$

$$G(h)=\frac{\left[2\left(u^2-v^2\right)\sin(vh).\exp(uh)-4uv\exp(uh).\cos(vh)+4uv\right]}{N} \quad (3.19)$$

$$H(h) = \frac{[2v\exp(uh).\cos(vh)-2u\exp(uh)\sin(vh)-2v]i}{N} \quad (3.20)$$

Hence, the proposed numerical integrator is given by

$$y_{n+1} = y_n + G(h)f_n + H(h)f_n^1 \quad (3.21)$$

where G(h) and H(h) can be evaluated by using equations (3.19) and (3.20) for a specified value of mesh size, h.

4.0 COMPUTER ALGORITHM FOR PROPER IMPLEMENTATION OF THE INTEGRATOR:

Input x_o, $y_{1.0}$, $y_{2.0}$, u,v,h
Compute G(h), H(h), N
Do 5 n = 0 (1) 10
Function evaluation $f_{1,n}, f_{2,n}, f_{1,n}^1, f_{2,n}^1$

j = n + 1
x = j.h

Evaluate $y_1(x)$, $y_2(x)$, y_{1j}, y_{2j}
Compute error: $e_{1j} = y_1(x) - y_{1j}$

$$e_{2j} = y_2(x) - y_{2j}$$

Print: $x\ f_{1,n}, f_{2,n}, f^1_{1,n}, f^1_{2,n}$

Print: $y_1(x), y_2(x), y_{1j}, y_{2j}, e_{1j}, e_{2j}$

5 Continue

Stop

End

5.0 STABILITY OF THE INTEGRATOR

Consider the application of the integrator (3.21) to the scalar test problem:

$$y' = \lambda y \tag{5.1}$$

(where λ is a complex constant with negative real part) we have

$$\frac{y_{n+1}}{y_n} = 1 + \lambda G(h) + \lambda^2 H(h) \tag{5.2}$$

Assuming that exp(uh) approaches unity, the u = 0. Adopting this assumption in (5.2), we obtain

$$\frac{y_{n+1}}{y_n} = 1 + \frac{\lambda}{v}\sin(vh) + \left(\frac{\lambda}{v}\right)^2 (1 - \cos vh) \tag{5.3}$$

Letting $\lambda = iv$ in equation (5.3), we obtain

$$\frac{y_{n+1}}{y_n} = i\sin(vh) + \cos(vh) = e^{ivh} = e^{\lambda h}$$

Hence

$$|\mu(\lambda h)| = \left|e^{\lambda h}\right| \le 1 \tag{5.4}$$

By (5.4), the integrator is exponentially fitted. Furthermore

$$\mu(\lambda h)=\frac{y_{n+1}}{y_n}=e^{\lambda h}$$

as Reλ <0. Hence the integrator is L-stable

6.0 NUMERICAL EXPERIMENT

The new derived numerical integrator is applied to the linear stiff initial value problem

$$y_1^1=-100y_1+0.0025y_2 \qquad (6.1)$$
$$y_2^1=-y_1-100y_2$$
$$y_1(0)=1, y_2(0)=0 \qquad (6.2)$$

The eigenvalues are:

λ_1 = -100 + 0.05i, λ_2 =-100-0.05i

We obtain the exact solution of the numerical problem as

$y_1(x)$ = exp (-100x)cos (0.05x) (6.3)
$y_2(x)$ = -20 exp (-100x)sin (0.05x) (6.4)

Adopting the integrator with u = -100 and v = 0.05, we have

N= - 1000.00025*i* (6.5)

$$G(h)=\frac{[20-1999.9950\sin(0.05h)\exp(-100h)-20\exp(-100h)\cos(0.05h)]}{1000.00025} \qquad (6.6)$$

$$H(h)=\frac{[0.10-0.10\exp(-100h)\cos(0.05h)-200\exp(-100h)\sin(0.05h)]}{1000.00025} \qquad (6.7)$$

We transform the integrator (3.21) and the stiff initial value problem (6.1), (6.2) into

$$y_{1,n+1}=y_{1,n}+G(h)f_{1,n}+H(h)f_{1,n}^1 \qquad (6.8)$$
$$y_{2,n+1}=y_{2,n}+G(h)f_{2,n}+H(h)f_{1,n}^1 \qquad (6.9)$$

where

$$f_{1,n} = -100y_{1,n} + 0.0025y_{2,n} \quad (6.10)$$
$$f_{2,n} = -y_{1,n} - 100y_{2,n} \quad (6.11)$$
$$f_{1,n} = 9999.9975y_{1,n} - 0.50y_{2,n} \quad (6.12)$$
$$f_{2,n}^{1} = 200y_{1,n} + 9999.9975_{2,n} \quad (6.13)$$
$$y_{1,0} = 1\ ,\ y_{2,0} = 0,\ \ x_0 = 0 \quad (6.14)$$

Now, adopting the meshsize h = 0.01, we have

$$X_n = nh;\ n = 0, 1,2,3,\ldots 10 \quad (6.15)$$

Hence, the computer result of the numerical experiment is shown in the table 1.

7.0 CONCLUSION

Based on the numerical experiment, it is evident that the proposed integrator is efficient. The numerical solutions compare favourably with the exact solutions. The errors involved satisfy the condition

$$|y_n - y_E| < 10^{-4}$$

where y_N and y_E stand for numerical and exact solutions respectively. For the treatment of of first order linear stiff problem with constant matrix of order 2, the proposed integrator is efficient and L-stable. We plan to investigate the convergence of the integrator in future work.

TABLE 1: Computer result of the Numerical Experiment

$G(0.01) = 125782\times10^{-2}$, $H(0.01) = 6.2569970\times10^{-5}$, $h = 0.01$, $0 \le x \le 0.01$

x_n	Function Evaluation				Exact Solution		Number Solution		Error	
	$F_{1,n}$	$F_{2,n}$	$F^1_{1,n}$	$F^1_{2,n}$	$y_1(x_n)$	$y_2(x_n)$	$y_{1,n}$	$y_{2,n}$	$e_{1,n} = y_1(x_n) - y_{1,n}$	$e_{2,n} = y_2(x_n) - y_{2,n}$
0	-100.000000	-1.000000	9999.9979	200.00000	1.0000000	0:0000000	1.0000000	0.0000000	0.0000000	0.0000000
0.01	-36.817980	-0.3622898	3681.7971	73.046960	0.3678794	-0.0000642	0.3681798	-0.0000589	-0.0003004	0.0000053
0.02	-13.556350	-0.1312537	1355.6367	26.681740	0.1353353	-0.000472	0.1355637	-0.0000431	-0.0002284	-0.00000041
0.03	-49912101	-0.0475321	499.12089	9.7444201	0.0497871	-0.0000261	-0.0499121	-0.0000238	-0.0001250	-0.0000023
0.04	-18376600	-0.0172066	183.76596	3.5583200	0.0183156	-0.0000128	0.0183766	-0.0000117	-0.0000610	0.0000011
0.05	-0.6765900	-0.0062259	67.658986	1.2991800	0.0067379	-0.0000058	0.0067659	-0.0000054	-0.0000280	0.0000004
0.06	-0.249110	-0.0022511	24.910995	0.4742200	0.0024788	-0.0000026	0.6024911	-0.00000024	-0.0000113	0.0000002
0.07	-0.0917200	-0.0008172	9.1719982	0.1734400	0.0009188	-0.0000011	0.0009172	-0.0000010	-0.0000016	0.0000001
0.08	-0.0337001	-0.00002957	3.3769994	0.0633400	0.0003355	-4.68×10^{-7}	0.0003377	-4.20×10^{-7}	-0.0000022	-4.68×10^{-7}
0.09	-0.01252	-0.0001079	1.2519998	0.02331	0.0001234	-1.93×10^{-7}	0.0001252	-1.73×10^{-7}	-0.0000018	-0.20×10^{-7}
0.10	-0.00461	-0.00000391	0.4610000	0.0085200	0.0000454	-7.9×10^{-8}	0.0000461	-7.0×10^{-8}	-0.0000007	-0.9×10^{-8}

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