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AN EXTENDED CONJUGATE GRADIENT ALGORITHM FOR DISCRETE OPTIMAL CONTROL PROBLEMS

FRANCIS O. OTUNTA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BENIN, BENIN CITY, NIGERIA.

ABSTRACT

Discrete Optimal Control Problems arise in many multistage and scheduling problems. In principle also, continuous optimal control problems may be discretized appropriately and subsequently be formulated as discrete optimal control problems. Herein, we propose an algorithm, based on the conjugate Gradient method for solving discrete optimal control problems with constraints on the states and controls of the dynamical system.

1. INTRODUCTION

Many efficient computational algorithms are now available in the literature for solving continuous optimal control problems, such as Gradient Methods: Miele [16], Ibiejugba and Onumanyi [10], Ladson et al [14], Damoulakis [4], Miele et al [17] etc; the multiplier Methods; Ibiejugba [11],, Ibiejugba et al [12], Miele et al [18] and Di Pillo et al [5] etc; the Gradient restoration methods: Miele and Wang [19], Miele [16] and Pritchard [26]; the Control parameterization methods: Goh and Teo [9], and the Factorization methods: Milman and Schied [20] and Milman and Schumitzky [21]. Thus the discretization of continuous optimal control problems seems to be no longer warranted.

Nevertheless, there remains much interest in the numerical solutions of the generically discrete optimal control problems. The fact that the continuous optimal control problems require us to determine measurable functions, which cannot be generated by digital computers makes the search for numerical solutions to discrete optimal control problems worthwhile.

Many methods are now available in the literature for solving such problems. Loosely, the available methods for Discrete Optimal Control Problems can be grouped into four main classes:

(a) Methods based on the maximum principles: Boltyanskii [2], Dolezal [6];

(b) Methods based on the Mathematical programming techniques; Cannon et al [3], Polak [24], etc;

(c) Methods based on a hybrid of the maximum principle and Mathematical programming techniques, Evtushenko [8], Mehre and Davis [15], Polak [24], Polyak [25], Teo et al [27]; and

(d) Methods using dynamic programming techniques, Bellman and Dreyfus [1], Dyer and Mc Reynolds [7], Jacobson and Mayne [13], Morin [22], Ohon [23], Yakowitz and Rutherford [28].

In this paper, we propose an algorithm, based on the conjugate gradient method for solving discrete optimal control problems with constraints on the states and controls of the dynamical system. It is important to note that, apart from the first class of solution techniques mentioned above, all the others employ at least a bit of mathematical programming. This is basically due to the relationship between Nonlinear programming problems and optimal control problems. We shall start by showing that, ordinarily, the Discrete Optimal Control problem can be viewed as a Nonlinear programming problem with some special structures.

The Nonlinear programming problem can generally be stated as follows:

Problem (P1)

Given continuously differentiable functions $f: R^n \rightarrow R^n$, $g: R^n \rightarrow R^m$ and $r: R^n \rightarrow R^q$, find a $\hat{z}$ in the set $\Omega = \{ z: g(z) \leq 0, r(z) = 0\}$, such that for all $z \, \varepsilon \Omega$, $f(z) \leq f(\hat{z})$.

We note here that the functions g and r have components $g_1, g_2, \ldots, g_m$ and $r_1, r_2, \ldots, r_q$ respectively. Thus, statements about g and r hold componentwise.

Problem (P1) can, however, be stated in shorthand form as:

$$\text{Minimize } \{f(z): g(z) \leq 0, r(z) = 0\}. \qquad (1.1)$$

The need for problem (P1) in this paper will become obvious when our problem of interest (the Discrete Optimal Control Problem) is shown to be a form of Problem (P1) with special structure. This is the concern of the next section.

2. PROBLEM STATEMENT

Consider a dynamical system, described by the system of difference equations

$$x_{i+1} = g_i(x_i, u_i),\ i = 0, 1, \ldots, k,\ x(0) = x_0 \qquad (2.1)$$

where $x = (x_1, x_2, \ldots, x_n)^T$ and $u = (u_1, u_2, \ldots, u_m)^T$ are respectively the state and control vectors; $g_i: R^n \times R^m \rightarrow R^n$ is a continuously differentiable function and $x_0 = (x_{1,0}, x_{2,0}, \ldots, x_{n,0})^T \, \varepsilon R^n$ is a given initial vector and $x_{i,0}$ stands for the i-th component of x_0.

Problem (P2)

Given the system (2.1), find a control $\hat{U} \varepsilon R^m$ and a corresponding trajectory $\hat{x} \, \varepsilon \; R^n$, such that the cost functional

$$J(x, u) = \sum_{i=0}^{k} f_i(x_i, u_i), \qquad (2.2)$$

is minimized over a class of all feasible control and state vectors, where $f_i: R^n \times R^m \rightarrow R$, is continuously differentiable and k is the duration of control process.

Problem (P1) is clearly the simpler and more studied of the two stated problems. Not surprisingly, therefore, the largest fraction of existing algorithms deal with Nonlinear programming problems.

Now, we consider problem (P2) and let

$$z = (x_0, x_1, \ldots, x_k, u_0, u_1, \ldots, u_k)^T \qquad (2.3)$$

$$f(z) = \sum_{i=0}^{k} f_i(x_i, u_i), \qquad (2.4)$$

$$r(z) = x_{i+1} - g_i(x_i, u_i), \quad i = 0, 1, \ldots, k. \qquad (2.5)$$

Then, it is not difficult to see that the Discrete Optimal Control Problem (P2) assumes the form of Problem (P1), without the inequality constraints. Thus, it becomes clear that, at least in principle, all nonlinear programming algorithms are applicable to the Discrete Optimal Control Problems. However, the very high dimensionality of the vector z in Equation (2.3) makes the transcription of the Discrete Optimal Control Problem to the Nonlinear programming form, unsatisfactory. Other forms of transcription are available, where for instance we let

$$z = (x_0, u_0, \ldots, u_k) \text{ and } x_i(x_0, u)$$

denote the solution of Equation (2.1) at the ith step, corresponding to $u = (u_0, u_1, \ldots, u_k)$. Although the nonlinear programming problem obtained via this scheme substantially reduces the dimension of the vector r(z), the story remains the same as we still have in our hands, a large dimensional problem to solve. Polak [24] contains other forms of transformation.

As a general rule, we may be better off, formulating the original control problem in discrete form when we are interested in on-line control of dynamical systems by means of a small computer, since in such cases the solution of continuous optimal control problems may not be practicable.

From the foregoing, our discrete optimal control problem of the form (P2) are usually encountered in constrained form and penalty functions are often used to cope with the constraints on the states and controls of the dynamical system.

In the next section, we put our problem of interest in a form suitable for the application of the Extended Conjugate Gradient Method, by constructing a matrix operator H, which proves useful in our subsequent developments.

3. CONSTRUCTION OF THE MATRIX OPERATOR H.

Our intent here, is to attempt to provide a direct numerical solution to the linear quadratic optimal control problem (LQOCP), given by Equation (2.2), subject to discrete time linear dynamical equation over the set, R, of real numbers.

For a start, we consider the one-dimensional case of Problem (P2). Thus our problem of interest is

Problem (P3)

$$\text{Minimize} \sum_{i=1}^{k} (ax_i^2 + bu_i^2) \tag{3.1}$$

Subject to

$$x_i = cx_{i-1} + du_{i-1},\ x_0 \text{ given}, \tag{3.2}$$

where a, b, c and d are constants with a, b, > 0, c and d are not necessarily positive but nonzero.

The introduction of a penalty constant $\mu(\mu > 0)$ converts the constrained problem (P3) to an unconstrained problem given by

Problem (P4)

$$Min\, J(x, u) = \text{Minimize} \left\{ \sum_{i=1}^{k} ax_i^2 + bu_i^2 + \mu[x_i - cx_{i-1} - du_{i-1}]^2 \right\} \tag{3.3}$$

If the problem can further be written in the form < Z, HZ>, for a given matrix operator H, and for the ordered pair $z_i = (x_i, u_i)^T$, then the Extended Conjugate Gradient Method due to Ibiejugba and Onumanyi [10] can be applied to solve the problem.

Our concern now is to construct the required matrix operator H, which renders problem (P4) amendable to the ECGM algorithm. We start by associating problem (P4) with the matrix operator H as follows.

$$< Z, HZ >_W = \sum_{i=1}^{k} \left[ax_i^2 + bu_i^2 + \mu(x_i - cx_{i-1} - d\, u_{i-1})^2 \right] \tag{3.4}$$

where W is a real Hilbert space and $<.,.>_W$ denotes the inner product in W, which in this case is taken to be the Euclidean space, and

$$Z = (x_0, x_1, \ldots, x_k, u_0, u_1, \ldots, u_k)^T. \tag{3.5}$$

Without much difficulty, we obtain

$$H = \begin{pmatrix} F & : & B \\ \cdots & : & \cdots \\ B^T & : & D \end{pmatrix} \tag{3.6}$$

Where F,B, and D are block matrices, whose entries are defined as follows:
F is a tridiagonal matrix of order k + 1, with entries given by;

$f_{ij} = -\mu c$ provided $|i-j| = 1$, $\underset{\substack{j\neq 1 \\ j\neq k+1}}{f_{jj}} = a + \mu(1+c)$,

$f_{11} = \mu c^2$, $f_{k+1,k+1} = a + \mu$ and $f_{ij} = 0$ otherwise.

B is a square matrix of order k + 1, with $b_{ij} = -\mu d$ $\forall i,j$ such that $i = 1 + j$,

$\underset{j\neq k+1}{b_{jj}} = \mu cd$, $b_{ij} = 0$ otherwise.

D is a diagonal matrix of order k + 1 with

$d_{11} = \mu d^2$, $d_{k+1,k+1} = b$ and $\underset{\substack{j\neq 1 \\ j\neq k+1}}{d_{jj}} = b + \mu d^2$.

As usual, B^T denotes the transpose of B. It is then very easily seen that
$<Z, HZ> = Z^T HZ$,
with Z and H given in Equations (3.5) and (3.6) respectively, restores the right hand side of Equation (3.4)

4. AN EXTENDED CONJUGATE GRADIENT METHOD FOR THE DISCRETE OPTIMAL CONTROL PROBLEM

The philosophy of the ECGM algorithm, due to Ibiejugba and Onumanyi [10] would be extended to cope with the minimization of problem (P4).

Let P_i denote the vector of the descent direction at the ith step of the ECGM algorithm, then using the matrix operator H, constructed in the previous section, we obtain HP_i as follows:

$$
HP_i = \begin{pmatrix}
\mu c^2 P_{x0} - \mu c P_{x_1} + \mu c\, dP_{u0} \\
- \mu c P_{x0} + (a + \mu(1+c)) P_{x_1} - \mu c P_{x2} - \mu d P_{u0} + \mu c\, dP_{u_1} \\
- \mu c P_{x_1} + (a + \mu(1+c)) P_{x2} - \mu c P_{x_3} - \mu d P_{u_1} + \mu cd P_{u2} \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
- \mu c\, dP_{x_{i-1}} + (1+\mu) P_{x_i} - \mu d P_{u_i - 1} \\
\mu cd P_{x0} - \mu d P_{x_i} + \mu d^2 P_{u0} \\
\mu cd P_{x_1} - \mu d P_{x2} + (b + \mu d^2) P_{u_1} \\
\vdots \quad \vdots \quad \vdots \\
\mu cd P_{x_{i-1}} - \mu d P_{x_i} + (b + \mu d^2) P_{u_i} \\
bP_{u_i}
\end{pmatrix} \qquad (4.1)
$$

for i = 1, 2, 3, ..., k.

Furthermore we define the descent step size at the ith step as

$$
\alpha_i = \frac{< g_i, g_i >}{< P_i, HP_i >} \qquad (4.2)
$$

where the step size between the controls are given by

$$
\alpha_{u_i} = \frac{< g_{u_i}, g_{u_i} >}{< P_{u_i}, HP_{u_i} >} \qquad (4.3)
$$

and similarly we have

$$
\alpha_{x_i} = \frac{< g_{x_i}, g_{x_i} >}{< P_{x_i}, HP_{x_i} >} \qquad (4.4)
$$

for the states vectors.

By virtue of Equations (4.1), (4.2), (4.3) and (4.4), we construct the Extended Conjugate Gradient Methods suitable for solving the control problem (P4). The algorithms are as follows:

Algorithm (4.1)
Step 1:

Select x_0, u_0 arbitrarily from the domain of the given problem, and compute the remaining members of the paired descent sequence with the aid of the formulas described by subsequent steps of the algorithm.

Step 2:

Compute the partial derivatives of J. Equation (3.3), with respect to x_i and u_i respectively for all i, viz:

$$g_{x_i} = \Delta J_{x_i}, g_{u_i} = \Delta J_{u_i}$$

9a nd set $\quad i = 0, 1, ,2, ..., k \qquad (4.5)$

$$P_{x_0} = - g_{x_0}, P_{u_0} = - g_{u_0}$$

Step 3:

Compute

$$x_{i+1} = x_i + \alpha_{x_i} P_{x_i}, u_{i+1} = u_i + \alpha_{u_i} P_{u_i} \qquad (4.6)$$

$$P_{x_{i+1}} = - g_{x_i} + \frac{< g_{x_{i+1}}, g_{x_{i+1}} >}{< g_{x_i}, g_{x_i} >} P_{x_i} \qquad (4.7)$$

$$P_{u_{i+1}} = - g_{u_i} + \frac{< g_{u_{i+1}}, g_{u_{i+1}} >}{< g_{u_i}, g_{u_i} >} P_{u_i}, \qquad (4.8)$$

where α_{u_i} a nd α_{x_i} are given by Equations (4.3) and (4.4) respectively, and HP_i given by Equation (4.1) with

$$HP_{x_i} = HP_i |_{P_{u_i} = 0} \qquad (4.9)$$

and

$$HP_{u_i} = HP_i |_{P_{x_i} = 0} \qquad (4.10)$$

Step 4:

Stopping Criterion: Terminate iteration if $g_{x_i} = 0$ o r/ a nd $g_{x_i} = 0$, otherwise stop when i = k, the specified duration of control process.

Clearly, Algorithm (4.1) is a discrete version of the ECGM originally Constructed for penalized cost functional for Continuous Optimal Control Problems. In what follows, we propose an iterative scheme for Computing x_i and u_i (i = 1, 2, 3, ..,k) from a first approximation, arbitrarily taken as in step 1 of Algorithm (4.1)

Algorithm (4.2)
Step 1: Same as step 1 of Algorithm (4.1)
Step 2: Same as step 2 of algorithm (4.1)
Step 3:
Compute

$$x_1 = x_0 = \frac{g_{x_0}}{\mu c^2}, u_i = u_0 = \frac{g_{u_0}}{\mu d^2} \tag{4.11}$$

$$x_{i+1} = x_i + \frac{<g_{x_i}, g_{x_i}><g_{x_{i-1}}, g_{x_{i-1}}>}{[a+\mu(1+c)]\left[<g_{x_i}, g_{x_i}>\left(-<g_{x_{i-1}}, g_{x_{i-1}}>\sum_{j=0}^{i-1}\frac{1}{g_{x_j}}\right)-<g_{x_{i-1}}, g_{x_{i-1}}>g_{x_1}\right]}$$

$$i = 2,3,\ldots,k. \tag{4.12}$$

and

$$u_{i+1} = u_i + \frac{<g_{u_i}, g_{u_i}><g_{u_{i-1}}, g_{u_{i-1}}>}{(b+\mu d^2)\left[<g_{u_i}, g_{u_i}>\left(-<g_{u_{i-1}}, g_{u_{i=1}}>\sum_{j=0}^{i-1}\frac{1}{g_{u_j}}\right)-<g_{u_{i-1}}, g_{u_{i-1}}>g_{u_1}\right]}$$

$$i = 2,3,\ldots,k. \tag{4.13}$$

Step 4: Same as step 4 of Algorithm (4.1).

Equations (4.11), (4.12) and (4.13) are obtained as follows: For i = 0, Equations (4.6) become

$$x_1 = x_0 + \alpha_{x_0} P_{x_0} \; and \; u_1 = u_0 + \alpha_{u_0} P_{u_0}. \tag{4.14}$$

Substituting Equation (4.5) into Equations (4.14) yields

$$x_1 = x_0 - \alpha_{x_0} g_{x_0} \text{ and } u_1 = u_0 - \alpha_{u_0} g_{u_0}.$$

But

$$\alpha_{x0} = \frac{<g_{x0}, g_{x0}>}{<P_{x0}, HP_{x0}>} = \frac{<g_{x0}, g_{x0}>}{<g_{x0}, Hg_{x0}>} = \frac{1}{\mu c^2}. \tag{4.15}$$

Hence

$$x_1 = x_0 - \frac{g_{x0}}{\mu c^2}$$

and

$$u_1 = u_0 - \frac{g_{u0}}{\mu d^2},$$

from the fact that

$$\alpha_{u0} = \frac{<g_{u0}, g_{u0}>}{<P_{u0}, HP_{u0}>} = \frac{<g_{u0}, g_{u0}>}{<g_{u0}, Hg_{u0}>} = \frac{1}{\mu d^2}.$$

For i = 1, Equations (4.6) become

$$x_2 = x_1 + \alpha_{x1} P_{x1} \quad \text{and} \quad u_2 = u_1 + \alpha_{x1} P_{u1}.$$

Using equation (4.1), with i = 1, we readily obtain

$$<P_{x1}, HP_{x1}> = [a + \mu(1+c)] P_{x1}^2,$$

so that

$$\alpha_{x1} P_{x1} = \frac{<g_{x1}, g_{x1}>}{[a + \mu(1+c)] P_{x1}}. \tag{4.16}$$

Using Equation (4.17) in Equation (4.16) and simplifying, we obtain

$$\alpha_{x1} P_{x1} = \frac{<g_{x1}, g_{x1}><g_{x0}, g_{x0}>}{[a + \mu(1+c)]\left[-g_{x1} <g_{x0}, g_{x0}> - <g_{x1}, g_{x1}> g_{x0}\right]}. \tag{4.17}$$

Hence

$$x_2 = x_1 + \alpha_{x1} P_{x1}$$

becomes

$$x_2 = x_1 + \frac{<g_{x_1}, g_{x_1}> <g_{x_0}, g_{x_0}>}{[a+\mu(1+c)]\left[-g_{x_1} <g_{x_0}, g_{x_0}> - <g_{x_1}, g_{x_1}> g_{x_0}\right]}. \quad (4.18)$$

Similarly, we obtain

$$u_2 = u_1 + \frac{<g_{u_1}, g_{u_1}> <g_{u_0}, g_{u_0}>}{[b+\mu d^2]\left[-g_{u_1} <g_{u_0}, g_{u_0}> - <g_{u_1}, g_{u_1}> g_{u_0}\right]}. \quad (4.19)$$

Simple inspection shows that for i = 2, Equations (4.12) and (4.13) reduce to Equations (4.18) and(4.19) respectively. For i > 2, we follow the same lines of argument as shown above and the principle of mathematical induction to verify Equations (4.12) and (4.13) for each i.

5. NUMERICAL EXPERIMENTS

In this section, two numerical examples are solved to illustrate Algorithm (4.2) proposed in section 4 of this paper. For each of the examples, we compute and report the values of the objective functional (cost functional) for indicated values of k (the duration of control process) and some penalty constants (arbitrarily chosen).

Example 1

Consider the following first-order system $x_i = 0.5x_{i-1} + u_{i-1}$, $x(0) = 1$ where i = 1, 2, .., k (k specified). The cost functional

$$OBJ = \sum_{i=1}^{k} (x_i^2 + u_i^2)$$

is to be minimized with the control bounded by $-1 \le u_i \le 1$, for i = 0, 1, 2, ..., k.

This problem had been considered by Teo et al [27], who solved the problem by employing some even and uneven parametrization partitions before introducing a simple constraint transcription to approximate each of the all-step constraints into a single constraint, cast in a canonical form similar to that of the cost functional, to simplify computation of gradients.

Table (4.1) depicts the numerical results obtained via our scheme (algorithm (4.2)), while Tables (4.1.1) and (4.1.2) show the results of Teo et al [27].

Table (4.1)

Numerical Results For Example 1 via Algorithm (4.2)

Value of k	OBJ Value	Penalty Constant (μ)
1	0.472222	Independent of μ
2	0.81720134	0.5
2	0.97984404	0.05
4	1.15931778	0.5
4	1.22022389	0.05

Table (4.1.1)

Numerical Results For Example 1 via an even Parametrization Partition (Teo et al [27])

No of Iterations	OBJ Value	No of Steps (Np)
10	1.2934773	5
16	1.2484346	10
31	1.1327822	50

Table (4.1.2)

Numerical Results For Example 1 via an Uneven Parametrization Partition (Teo et al [27])

No of Iterations	OBJ Value	No of Steps (Np)
15	1.1327825	10

Example 2.

$$\text{Minimize} \sum_{i=0}^{k} x_i^2 + u_i^2$$

subject to

$$x_i - x_{i-1} = u_{i-1};\ x_0 = 1.$$

Algorithm (4.2) is used, with initial guess of u(0) = 0.5 for the optimal control. Table (4.2) shows the numerical results obtained for k = 1, 2 with the parameter 1.0, 0.5, 0.25 and 10.

Table (4.2)

Numerical Results For Example 2 via Algorithm (4.2)

Value of k	OBJ Value	Penalty Parameter (μ)
1	10.25	Independent of μ
2	16.10030511	1.0
2	16.24449598	0.5
2	16.36777588	0.25
2	15.84573272	10

5. CONCLUSION

We have, in section 3 of this paper given a scheme for determining the matrix operator H, for any Discrete Optimal Control Problem of the type given by problem (P2). Furthermore, we have given expressions for the direct computation of the state and control variables for this class of problems.

Although no theoretical results are available for comparison, the results published by Teo et al [27], indicate that an OBJ value of 1.1327825 is quite good for our Example 1. This is based on the fact that Teo et al [27] accepted that an uneven parametrization partition is more expedient, as it reduces significantly, the required computational effort in solving Example 1. Table (4.1) shows that the results obtained via our Algorithm (4.2) are not out of place.

It is important to note that the determination of the state variables via our proposed scheme involves implicit equations that may not be solved accurately. Solutions of such equations may be achieved via any convergent one-point iterative scheme. The Newton-Raphson scheme is recommended here.

For better approximations, we require higher values of k. This is clearly revealed by Table (4.1). However, the use of high values of k leads to the manipulation of very large matrices. This snag is currently being investigated and is one of the points to be reported in a forthcoming paper, where a generalization of problem (P3) is also given.

In conclusion, we note that our scheme is a penalty function method, and so largely depends on the choice of the penalty parameter. From the foregoing, however, it is observed that for k = 1 our scheme is independent of the penalty parameter. Unfortunately, the value of the cost functional at k = 1 remains the worst approximation. This again, is clearly seen from the results shown in Table (4.1).

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