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AN OPTIMIZATION TECHNIQUE FOR THE RETARDED DIFFERENTIAL DELAY EQUATION

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ABSTRACT

Models in natural settings sometime include the present and also the past states of the system. This is natural in application in many branches of human endeavour. In this work, the minimization of a quadratic cost functional subject to the retarded differential delay equation is presented. The elegant algorithm of the Extended Conjugate Gradient Method (ECGM) originated in [1] was used.

1. INTRODUCTION

Differential equations which take into account the present as well as the past states of a physical systems are called Delay or Lag Differential equations.

This equations are in general of two types. Firstly, there are those with delays in the states of the systems called Retarded Differential equations. An example of this type, for instance in E^n, the n dimensional Euclidean space can be written as follows:

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-r)), \quad t \geq 0$$
$$x(0) = x_0 = \phi, \quad t \in [-r,0]. \qquad (1.1)$$

Retarded differential equations have for some time now been the object of active research. Interest in such equations result from the fact that delay differential equations express models more in their natural settings and so they admit fewer assumptions than ordinary differential equations. Moreover, methods of ordinary differential equations do not easily carry over to delay differential equations. As one would expect, problems associated with delay differential equations are in general more complicated than those associated with ordinary differential equations.[2]

The second type are those with delays in the states of the systems and also in the derivatives called Neutral Differential Equations. In E^n, an equation of the Neutral type can be given in the form:

$$\frac{d}{dt}[x(t) - Ax(t-r)] = f(t, x(t), x(t-r))$$
$$x(s) = \phi(s), \quad s \in [-r, 0], \qquad (1.2)$$

where, A is a non-zero constant n x n matrix. Observe that if A is the zero matrix, the neutral system (1.2) becomes the retarded differential equation (1.1).

The research efforts in the main has been geared towards obtaining adequate criteria for stability, controllability and observability of the system Our aim in this work is to extend the analysis of Reju and Ibiejugba [3] in the application of the ECGM algorithm to the control problem for systems governed by differential delay equations. The same approach was followed by Aderibigbe [4] in his work on the application of the ECGM algorithm to the control of differential delay equations.

However, in this work, attempt has been made to retain our resulting differential equation in their partial forms and get them solve analytically in their time and space variables in the construction of the ECGM control operator. This therefore results in having our control operator retaining the original two independent time (t) and space (x) variables of the delay equations.

PROBLEM (P1):

$$\text{Min } J(Z,U) = \text{Min} \int_0^1 \int_0^1 [U^2(x,t) + Z^2(x,t)]\, dx dt$$

Subject to

$$\frac{\partial z(x,t)}{\partial t} = C_1 Z(x,t) + C_2 Z(x,t-r) + dU(x,t), \quad 0 \le x, t \le 1$$
$$Z(x,t) = h(x,t); \qquad -r \le t \le 0$$

Where the delay parameter $r > 0$, is given; C_1, C_2 and d are specified constants which are not necessarily positive and h(x,t) is a given piecewise continuous function which is of exponential order on [-r, 0].
The constrained minimization problem (P1) is transformed into an unconstrained equation;

PROBLEM (P2):

$$\underset{(Z,U)}{\mathrm{Min}}\, J(Z,U,\mu) = \underset{(Z,U)}{\mathrm{Min}} \int_0^1\int_0^1 [\, U^2(x,t) + Z^2(x,t)]\, dxdt +$$

$$\mu \int_0^1\int_0^1 \| \frac{\partial z}{\partial t}(x,t) - C_1 Z(x,t) - C_2 Z(x,t-r) - dU(x,t) \|^2\, dxdt$$

where $\mu > 0$, is the penalty constant parameter.

We write (P2) in a bilinear form as:

PROBLEM (P3):

$$\underset{W}{\mathrm{Min}}\ J(W_1, W_2) = \underset{W}{\mathrm{Min}} < W_1, AW_2 >_H :\ W = (W_1, W_2)$$

$$= \underset{(Z,U)}{\mathrm{Min}} \Big\{ \int_0^1\int_0^1 [\, U_1(x,t)U_2(x,t) + Z_1(x,t)Z_2(x,t) + \mu Z_{1t}(x,t)Z_{2t}(x,t)$$

$$+ C_1^2 Z_1(x,t)Z_2(x,t) + \mu C_2^2 Z_1(x,t-r)Z_2(x,t-r) \quad + \mu d^2 U_1(x,t)U_2(x,t)$$

$$- \mu C_1 Z_{1t}(Z_2(x,t) - \mu C_1 Z_{2t}(x,t)Z_1(x,t) - \mu C_2 Z_{1t}(x,t)Z_2(x,t-r)$$

$$+ \mu d^2 U_1(x,t)U_2(x,t) - \mu C_1 Z_{1t}(Z_2(x,t) - \mu C_1 Z_{2t}(x,t)Z_1(x,t) - \mu C_2 Z_{1t}(x,t)Z_2(x,t-r)$$

$$- \mu C_2 Z_{2t}(x,t)Z_1(x,t-r) - \mu d Z_{1t}(x,t)U_2(x,t) - \mu d Z_{2t}(x,t)U_1(x,t) + \mu C_1 C_2 Z_1(x,t)Z_2(x,t-r)$$

$$+ \mu C_1 C_2 Z_2(x,t)Z_1(x,t-r) + \mu C_1 d Z_1(x,t)U_2(x,t) \quad + \mu C_1 d Z_2(x,t)U_1(x,t)$$

$$+ \mu C_2 d Z_1(x,t-r)U_2(x,t) + \mu C_2 d Z_2(x,t-r)U_1(x,t)] \Big\}$$

$$= \underset{W}{\mathrm{Min}} \int_0^1\int_0^1 W^T(x,t) A W(x,t)\, dxdt$$

where:

$$W(x,t) = (Z(x,t), U(x,t), Z_t(x,t), Z(x,t-r))$$

$$Z(x,t) = (Z_1(x,t), Z_2(x,t))$$

$$U(x,t) = (U_1(x,t), U_2(x,t)),$$

and the associated control operator A in (P3) is given by

$$\begin{pmatrix} 1+\mu d^2 & \mu C_1 d & -\mu d & \mu C_2 d \\ \mu C_1 d & 1+\mu C_1^2 & -\mu C_1 & \mu C_1 C_2 \\ -\mu d & -\mu C_1 & \mu & -\mu C_2 \\ \mu C_2 d & \mu C_1 C_2 & -\mu C_2 & \mu C_2^2 \end{pmatrix}$$

2. DERIVATION OF THE ECGM FUNDAMENTAL DIFFERENTIAL EQUATION

To construct the ECGM control operator, we associate with (P3) a new operator B satisfying

$$< W_1(x,t), BW_2(x,t) >_H$$

$$= \int_0^1\int_0^1 [U_1 U_2 Z_1 Z_2 + \mu Z_{1t} Z_{2t} + \mu C_1^2 Z_1 Z_2 + \mu C_2^2 Z_1(x,t-r) Z_2(x,t-r) + \mu d^2 U_1 U_2$$

$$- \mu C_1 Z_{1t} Z_2 - \mu C_2 Z_{2t} Z - \mu C_2 Z_{1t} Z_2(x,t-r) - \mu C_2 Z_{2t} Z_1(x,t-r) - \mu d Z_{1t} U_2$$

$$- \mu d Z_{2t} U_1 + \mu C_1 C_2 Z_1 Z_2(x,t-r) + \mu C_1 C_2 Z_2 Z_1(x,t-r)$$

$$+ \mu C_1 d Z_1 U_2 + \mu C_1 d Z_2 U_1 + \mu C_2 d Z_1(x,t-r) U_2 + \mu C_2 d Z_2(x,t-r) U_1]dxdt \qquad (2.1)$$

where:

$$Z_1 = Z_1(x,t),\ Z_2 = Z_2(x,t),\ Z_{1t} = Z_{1t}(x,t),\ Z_{2t} = Z_{2t}(x,t)$$

and H = $W_2^1[0, 1] \times L_2[0, 1] \times L_2[-r, 0]$, where $W_2^1[0, 1]$ is the Sobolev space of absolutely continuous functions $\chi(.)$ square integrable over [0, 1], while $L_2[\alpha,\beta]$ is the Hilbert space of equivalence classes of real valued functions on $\{\alpha, \beta\}$.

If $W(x, t) \in H$ denote the triplet

$$W^T(x,t) = (Z(x,t), U(x,t), h(x,t));\ Z(x,t) \in H[0, 1],$$
$$U(x,t) \in L_2[0, 1],\ h(x,t) \in L_2[-r, 0].$$

In (2.1), we set

$$W_2 = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} \overline{Z}_2 \\ \overline{U}_2 \\ \overline{h}_2 \end{pmatrix}$$

First, we obtain B_{11}, B_{21} and B_{31} by setting $\overline{U}_2 \equiv \overline{h}_2 = 0$, and note that when $h_2 = 0$,

then $Z_2(x, t-r) = Z_2(x, t)$. This gives

$$\begin{pmatrix} B_{11}Z_2 \\ B_{21}Z_2 \\ B_{31}Z_2 \end{pmatrix} = \begin{pmatrix} \overline{Z}_{11} \\ \overline{Z}_{21} \\ Z_{31} \end{pmatrix}$$

$$= \int_0^1\int_0^1 [\, Z_1\{(1+\mu C_1^2)Z_2 - \mu Z_{2t}\} + Z_{1t}\{\mu Z_{2t} - (\mu C_1 + \mu C_2)Z_2\} + U_1\{\mu C_1 dZ_2 + \mu C_2 dZ_2 - \mu dZ_{2t}$$
$$+ Z_1(x, t-r)\{\mu C_1C_2Z_2 + \mu C_2^2 Z_2 - \mu C_2 Z_{2t}\}]dxdt \qquad (2.2)$$

But

$$\int_0^1\int_0^1 Z_1(x, t-r)\{(\mu C_2^2 + \mu C_1C_2)Z_2 - \mu C_2 Z_{2t}\}dxdt$$

$$= \int_0^1\int_{-r}^{1-r} Z_1(x, s)\{(\mu C_2^2 + \mu C_1C_2)Z_2(x, s+r) - \mu C_2 Z_{2t}(x, s+r)\}dsdx$$

$$= \int_0^1\int_{-r}^{0} Z_1(x, s)\{(\mu C_2^2 + \mu C_1C_2)Z_2(x, s+r) - \mu C_2 Z_{2t}(x, s+r)\}dsdx$$

$$+ \int_0^1\int_0^{1-r} Z_1(x, s)\{(\mu C_2^2 + \mu C_1C_2)Z_2(x, s+r) - \mu C_2 Z_{2t}(x, s+r)\}dsdx$$

$$= \int_0^1\int_{-r}^{0} h_1(x, t)\{(\mu C_2^2 + \mu C_1C_2)Z_2(x, s+r) - \mu C_2 Z_{2t}(x, t+r)dtdx$$

$$+ \int_0^1\int_0^{1-r} Z_1(x, t)\{(\mu C_2^2 + \mu C_1C_2)Z_2(x, t+r) - \mu C_2 Z_{2t}(x, t+r)dtdx$$

since $Z_1(x, t) = h_1(x, t)$ for $t \in [-r, 0]$.
Now, for $t \in [0, 1-r]$, let us define

$$V_2(x, t) = \begin{cases} Z_2(x, t+r) & 0 \le t \le 1-r \\ 0 \quad ; & 1-r \le t \le 1 \end{cases}$$

(2.2) becomes,

$$\int_0^1\int_0^1 [Z_1\{(1+\mu C_1^2)Z_2 - \mu C_1 Z_{2t}\} + Z_{1t}\{\mu Z_{2t} - (\mu C_1 + \mu C_2)Z_2\} + U_1\{\mu C_1 dZ_2 + \mu C_2 dZ_2$$

$$- \mu dZ_{2t}\} + U_1\{\mu C_1 dZ_2 + \mu C_2 dZ_2 - \mu dZ_{2t}\}$$

$$+ h_1\{(\mu C_2^2 + \mu C_1 C_2)Z_2(x,t-r) - \mu C_2 Z_{2t}(x,t-r)$$

$$+ Z_1\{(\mu C_2^2 + \mu C_1 C_2)V_2 - \mu C_2 V_{2t}\}]dxdt$$

or

$$\int_0^1\int_0^1 [Z_1\{(1+\mu C_1^2)Z_2 - \mu C_2^2 + \mu C_1 C_2)V_2$$

$$-\mu C_2 V_{2t}\} + Z_{1t}\{\mu Z_{2t} - \mu C_1 Z_2 - \mu C_2 Z_2\} \qquad (2.3)$$

$$+ U_1\{\mu C_1 dZ_2 + \mu C_2 dZ_2 - \mu dZ_{2t}\}$$

$$+ h_1\{(\mu C_2^2 + \mu C_1 C_2)Z_2(x,t+r) - \mu C_2 Z_{2t}(x,t+r)\}]dxdt.$$

Write (2.3) as

$$\int_0^1\int_0^1 [Z_1\{(1+\mu C_1^2)Z_2 - \mu C_1 Z_{2t} + (\mu C_2^2 + \mu C_1 C_2)V_2$$

$$- \mu C_2 V_{2t}\} + Z_{1t}\{\mu Z_{2t} - \mu C_1 Z_2 - \mu C_2 Z_2\}$$

$$+ U_1\{\mu C_1 dZ_2 + \mu C_2 dZ_2 - \mu dZ_{2t}\}$$

$$+ h_1\{(\mu C_2^2 + \mu C_1 C_2)Z_2(x,t+r) - \mu C_2 Z_{2t}(x,t+r)\}dxdt$$

$$= \int_0^1\int_0^1 [Z_1 \bar{Z}_{11} + Z_{1t}\bar{Z}_{11t} + U_1 \bar{Z}_{21} + h_1 \bar{Z}_{31}]dxdt. \qquad (2.4)$$

Using the result due to Gelfand and Fomin [5], in (2.4), we let

$$\alpha = (1+\mu C_1^2)Z_2 - \mu C_1 Z_{2t} + (\mu C_2^2 + \mu C_1 C_2)V_2 - \mu C_2 V_{2t} \qquad (2.5a)$$

$$\beta = \mu Z_{2t} - (\mu C_1 + \mu C_2)Z_2 \qquad (2.5b)$$

In that wise, from (2.4),

$(\alpha - \bar{Z}_{11})$, $(\beta - \bar{Z}_{11t})$ as a consequence of the result in [5], are continuous functions on $[0,1]\times[0,1]$ and are n - times continuously differentiable on $[0,1]\times[0,1]$ and the space is equipped with the norm

$$\|\chi\|_n = \max_{0 \le x,t \le 1} |\chi| + \max_{0 \le x,t \le 1} |\chi^{(1)}| + \ldots + \max_{0 \le x,t \le 1} |\chi^{(n)}|$$

Where $\chi^{(n)}$ is the nth - differential of $\chi = \chi(\xi)$,
$\xi = (x,t) \in [0,1]\times[0,1]$.

Thus, if

$$\int_0^1 \int_0^1 [Z_1(\alpha - \bar{Z}_{11}) + Z_{1t}(\beta - \bar{Z}_{1lt})dxdt = 0,$$

then $(\beta - \bar{Z}_{1lt})$ is continuoully partially differentible with respectto their variables

We have,

$$\frac{\partial}{\partial t}[\beta - \bar{Z}_{11t}] = \alpha - \bar{Z}_{11} \tag{2.6}$$

hence,

$$\beta_t - \bar{Z}_{11tt} = \alpha - \bar{Z}_{11} \tag{2.7}$$

or

$$\bar{Z}_{11tt} - \bar{Z}_{11} = \beta_t - \alpha.$$

If $\beta_t - \alpha = F(x,t)$, equation (2.7) becomes

$$\bar{Z}_{11tt} - \bar{Z}_{11} = F(x,t). \tag{2.8}$$

In order to obtain a unique solution of (2.8), we impose appropriate initial conditions and solve the arising equation in the next section.

3. ANALYTIC SOLUTION TO THE SECOND ORDER EQUATION PROBLEM (P4):

Solve:

$$\frac{\partial^2}{\partial t^2} Z(x, t) - Z(x, t) = F(x, t) \tag{3.1}$$

subject to:

$$Z(x, 0) = n_0,\ Z_t(x, 0) = m_0 \tag{3.2}$$

where, $0 \le x, t \le 1$

To solve (P4), we take the Laplace transform of (3.1) on t – space to obtain [6]

$$L\{ \frac{\partial^2}{\partial t^2} Z(x, t) \} = s^2 Z(x, s) - sZ(x, 0) - Z_t(x, 0)$$

Where: $L\{Z(x, t)\} = Z(x, s)$

Hence, (3.1) becomes

$$s^2 Z(x, s) - sZ(x, 0) - Z_t(x, s) = F(x, s) \tag{3.3}$$

Simplifying after using initial conditions (3.2), equation (3.3) becomes

$$(s^2 - 1)Z(x, s) = F(x, s) + s\eta_0 + m_0$$

or

$$Z(x, s) = \frac{F(x,s)}{s^2 - 1} + \frac{S\eta_0}{s^2 - 1} + \frac{m_0}{s^2 - 1} \tag{3.4}$$

Taking the inverse transform in t – space, we note that

$$L^{-1}\{Z(x,s)\} = Z(x, t)$$

$$L^{-1}\{\frac{s}{s^2 - 1}\} = \text{Cosht}$$

$$L^{-1}\{\frac{1}{s^2 - 1}\} = \text{Sinht}$$

$$L^{-1}\{\frac{F(x, s)}{s^2 - 1}\} = F(x, s) * \text{Sinht}$$

Where $F(x, t) * \text{Sinht}$ is the convolution of $F(x, t)$ and Sinht given by [7]

$$F(x, t) * \text{Sinht} = \int_0^t F(x, \tau) \text{Sinh}(t - \tau)\, d\tau$$

So that from (3.4), we have

$$Z(x,t) = \int_0^t F(x,\tau)d\tau + \eta_0 \text{Cosh}t + m_0 \text{Sinh}t \qquad (3.5)$$

For the integral on the right hand side of (3.5) the following remark is in order

REMARK (R1)

$$\int_0^t F(x,\tau)\ \text{Sin ht}(t-\tau)d\tau$$

$$= \int_0^t \{\beta_t - \alpha\}\text{Sinh}(t-\tau)d\tau$$

$$= \int_0^t \beta_t\ \text{Sinh}(t-\tau)d\tau - \int_0^t \alpha\,\text{Sinh}(t-\tau)d\tau$$

For the first integral, set $U = \text{Sinh}(t-\tau)d\tau$

and $dv = \beta_t$ in the formula

$\int_0^x udv = [UV]\Big|_0^x - \int_0^x vdu$. Thus we have

$du = \text{Cosh}(t-\tau)d\tau; \quad V = \beta$

Integral becomes, using the formula

$$\beta\,\text{Sinh}(t-\tau)d\tau\Big|_0^t - \int_0^t \beta\,\text{Cosh}(t-\tau)d\tau$$

$$= -\beta(x,0)\ \text{Sinh}t + \int_0^t \beta(x,\tau)\ \text{Cosh}(t-\tau)d\tau$$

$$- \int_0^t \alpha(x,\tau)\ \text{Sin h}(t-\tau)d\tau$$

From (2.5a) and (2.5b), we have

$$\int_0^t F(x,\tau)\ \text{Sinh}(t-\tau)d\tau$$

$$= [-(\mu Z_{2t}(x,0) + (\mu C_1 + \mu C_2)\ Z_2(x,0))]\ \text{Sinh}t$$

$$+ \int_0^t \{[\mu Z_{2t}(x,s) + (\mu C_1 + \mu C_2)\ Z_2(x,s)]\ \text{Cosh}(t-\tau)\}d\tau$$

$$-\int_0^t \{[(1+\mu C_1^2)Z_2(x,s)-\mu C_1 Z_{2t}(x,s)+(\mu C_2^2+\mu C_1C_2)V_2(x,s)$$
$$-\mu C_2 V_{2t}(x,s)]\text{Sinh}(t-\tau)\}d\tau \quad (3.6)$$

$$Z(x,t)=[-(\mu Z_{2t}(x,0)+(\mu C_1+\mu C_2)Z_2(x,0)]\text{Sinht}$$
$$+\int_0^t \{[\mu Z_{2t}(x,s)+(\mu C_1+\mu C_2)Z_2(x,s)]\text{Cosh}(t-\tau)d\tau$$
$$-\int_0^t \{[(1+\mu C_1^2)Z_2(x,s)-\mu C_1 Z_{2t}(x,s)$$
$$+(\mu C_2^2+\mu C_1C_2)V_2(x,s)-\mu C_2 V_{2t}(x,s)]\text{Sinh}(t-\tau)d\tau$$
$$+\eta_0\,\text{Cosht}+m_0\,\text{Sinht}. \quad (3.7)$$

We proceed to determine η_0 and m_0

Integrating by parts

$$\int_0^1\Big[\int_0^1 Z_1(x,t)[\alpha(x,t)-\bar{Z}_{11}(x,s)]dt\Big]dx$$
$$=\Big[Z_1(x,t)\int_0^t\big[\alpha(x,s)-\bar{Z}_{11}(x,s)\big]ds\Big| \quad -\int_0^1\Big\{\Big[Z_{1t}(x,t)\int_0^t\big[\alpha(x,\tau)-\bar{Z}_{11}(x,\tau)\big]d\tau\Big]\Big\}dt\Big]$$
$$=-1\int_0^1\{[Z_{1t}(x,t)\int_0^1[\alpha x(x,\tau)-\bar{Z}_{11}(x,\tau)]d\tau]\}dt$$

since

$$Z_1(x,0)=Z_1(x,l)=0$$

Applying the last result gives

$$\int_0^1\Big[\int_0^1\Big\{Z_1(x,t)\big[\alpha(x,t)-\bar{Z}_{11}(x,t)\big]dt+\int_0^1\Big\{Z_{1t}(x,t)\big[\beta(x,t)-\bar{Z}_{11t}(x,t)\big]\Big]\Big\}dt\Big]dx$$
$$=\int_0^1\Big(-\int_0^1\Big\{Z_{1t}(x,t)\int_0^t\big[\alpha(x,\tau)-\bar{Z}_{11}(x,\tau)\big]d\tau\Big\}dt+\int_0^1\Big\{Z_{1t}(x,t)\big[\beta(x,t)-\bar{Z}_{11t}(x,t)\big]\Big]dt\Big)dx$$
$$=\int_0^1\int_0^1\Big\{Z_{1t}(x,t)\Big[\Big\{\beta(x,t)-\bar{Z}_{11t}(x,t)\Big\}-\int_0^t\big[\alpha(x,\tau)-\bar{Z}_{11}(x,\tau)\big]d\tau\Big]\Big\}dtdx$$

which implies from (2.6) that

$$\alpha(x,1) - \overline{Z}_{11}(x,1) = 0 \qquad (3.8)$$

$$\alpha(x,0) - \overline{Z}_{11}(x,0) = 0. \qquad (3.9)$$

Now by (3.2), $\overline{Z}_{11}(x,0) = \eta_0$ so that from (3.8) and (3.9), we obtain

$$\eta_0 = \alpha(x,0) = (1+\mu C_1^2)Z_2(x,0) - \mu C_1 Z_{2t}(x,0) + (\mu C_2^2 + \mu C_1 C_2)V_2(x,0) - \mu C_2 V_{2t}(x,0) \qquad (3.10)$$

Similarly, by (3.8), (2.5a) and (3.5)

$$Z11(x,1) = \alpha(x,1) = (1+\mu C_1^2)Z_2(x,1) - \mu C_1 Z_{2t}(x,1) + (\mu C_2^2 + \mu C_1 C_2)V2(x,0) - \mu C_2 V_{2t}(x,0) \qquad (3.11)$$

$$= \int_0^1 F(x,s)\,\mathrm{Sinh}(1-s)\,ds + \eta_0 \cosh 1 + m_0 \sinh 1 \qquad (3.12)$$

and using (3.10) in (3.12), we have

$$m_0 = [\,(1+\mu C_1^2)Z_2(x,1) - \mu C_1 Z_{2t}(x,1) + (\mu C_2^2 + \mu C_1 C_2)V_2(x,1)) - \mu C_2 V_{2t}(x,1) - \int_0^1 F(x,s)\sinh(1-s)ds + \{(1+\mu C_1^2)Z_2(x,0) - \mu C_1 Z_{2t}(x,0) + (\mu C_1^2 + \mu C_1 C_2)V_2(x,0) - \mu C_2 V_{2t}(x,0)\}/\sinh 1$$

Substituting for $\int_0^1 F(x,s)\,\mathrm{Sinh}(1-s)\,ds$ from (3.6), we get

$$m_0 = [\,(1+\mu C_1^2)Z_2(x,1) - \mu C_1 Z_{2t}(x,1) + (\mu C_1^2 + \mu C_1 C_2)V_2(x,1)] - \mu C_2 V_{2t}(x.1) - \{[-\mu Z_{2t}(x,0) + (\mu C_1 + \mu C_2)Z_2(0,1)]\}]\,\mathrm{Sinh}\,1$$

$$+ \int_0^1 [\mu Z_2(x,1) + (\mu C_1 + \mu C_2)Z_2(x,1)]\,\mathrm{Cosh}(t-s)\,ds$$

$$- \int_0^1 [(1+\mu C_1^2)Z_2(x,1) - \mu C_1 Z_{2t}(x,1) + (\mu C_2^2 + \mu C_1 C_2)V_2(x,1) - \mu C_2 V_{2t}(x,1)]\,\mathrm{Sinh}(t-s)\,ds$$

$$+ \{(1+\mu C_1^2)Z_2(x,0) - \mu C_1 Z_{2t}(x,0) + (\mu C_2^2 + \mu C_1 C_2)V_2(x,0) - \mu C_2 V_{2t}(x,0)\}\cosh 1]/\mathrm{Sinh}1 \qquad (3.13)$$

Finally using (3.13) and recalling that

$V_2(x,t) = Z_2(x,t+r)$, we obtain by virtue of (3.5)

$$\bar{Z}_{11}(x,t) = B_{11} Z(x,t) = -[\mu Z_{2t}(x,0) - (\mu C_1 + \mu C_2) Z_2(x,0)]\text{Sinh}t$$
$$+\int_0^t \{[\mu Z_{2t}(x,s) + (\mu C_1 + \mu C_2) Z_2(x,s)]\text{Cosh}(t-s)\}ds$$

$$-\int_0^t \left\{ \left[\begin{matrix} (1+\mu C_1^2) Z_2(x,s) - \mu C_1 Z_{2t}(x,s) + (\mu C_2^2 + \mu C_1 C_2) Z(x,t+r) - \\ \mu C_2 Z_{2t}(x,s+r) \end{matrix} \right] \text{Sinh}(t-s) \right\} ds$$

$$+[(1+\mu C_1^2) Z_2(x,0) - \mu C_1 Z_{2t}(x,0) + (\mu C_2^2 + \mu C_1 C_2) Z(x,r) - \mu C_2 Z_{2t}(x,r)]\text{Cosh}t$$

$$+\left[\begin{matrix} (1+\mu C_1^2) Z(x,1) - \mu C_1 Z_{2t}(x,1) + (\mu C_2^2 + \mu C_1 C_2) Z_2(x,1+r) - \\ \mu C_2 Z_{2t}(x,1+r) - \{-\mu Z_{2t}(x,0) + (\mu C_1 + \mu C_2) Z_2(x,0)\} \\ + \int_0^1 [\mu Z_{2t}(x,1) - (\mu C_1 + \mu C_2) Z_2(x,1)]\text{Cosh}(1-s)ds \end{matrix} \right]$$
$$-\int_0^1 [(1+\mu C_1^2) Z_2(x,1) - \mu C_1 Z_{2t}(x,1) + (\mu C_2^2 + \mu C_1 C_2) Z_2(x,1+r) - \mu C_2 Z_{2t}(x,1+r)]\text{Sinh}(1-s)$$
$$+\left\{ \begin{matrix} (1+\mu C_1^2) Z_2(x,0) - \mu C_1 Z_{2t}(x,0) + (\mu C_2^2 + \mu C_1 C_2) Z(x,r) \\ -\mu C_2 Z_{2t}(x,r) \end{matrix} \right\} \text{Cosh}\,1/\text{Sinh}\,1$$

(3.14)

$$\bar{Z}_{21}(x,t) = B_{21} Z(x,t) = \mu C_1\, dZ(x,t) + \mu C_2\, dZ(x,t) - \mu\, dZ_{2t}(x,t) \qquad (3.15)$$

$$\bar{Z}_{31}(x,t) = B_{31} Z(x,t) = (\mu C_2^2 + \mu C_1 C_2) Z(x,t+r) - \mu C_1 Z_{2t}(x,t+r) \qquad (3.16)$$

4 SECOND 2ND ORDER EQUATION

In the process of obtaining B_{12}, B_{22} and B_{32} by setting $\bar{Z}_2 \equiv \bar{h}_2 = 0$, then $Z_{2t} = 0 = Z_2(x,t-r)$.

We thus have from the explicit expression given in (2.1)

$$\begin{bmatrix} B_{12} & U_2 \\ B_{22} & \overline{U}_2 \\ B_{32} & \overline{U}_2 \end{bmatrix} = \begin{bmatrix} \overline{Z}_{12} \\ \overline{Z}_{22} \\ \overline{Z}_{32} \end{bmatrix}$$

$$= \int_0^1 \int_0^1 \left[Z_1(\mu C_1 dU_2) + Z_{1t}(-\mu dU_2) + U_1\left(1+\mu d^2\right)U_2 + Z_1(x, t-r)(\mu C_2 dU_2)\right] dxdt \qquad 4.1$$

But

$$\int_0^1 \int_0^1 Z_1(x,t-r)[\mu C_2 dU_2] dxdt$$
$$= \int_0^1 \int_{-r}^{1-r} Z_1(x,s)[\mu C_2 dU_2(x,s+r)] dsdx$$
$$= \int_0^1 \int_{-r}^{0} Z_1(x,s)[\mu C_2 dU_2(x,s+r)] dsdx + \int_0^1 \int_0^{1-r} Z_1(x,s)[\mu C_2 dU_2(x,s+r)] dsdx$$
$$= \int_0^1 \int_{-r}^{0} h_1(x,t)[\mu C_2 dU_2(x,t+r)] dtdx + \int_0^1 \int_0^{1-r} Z_1(x,t)[\mu C_2 dU_2(x,t+r)] dtdx$$

Setting $P_2(x,t) = \begin{cases} U(x,t+r) & ; \quad 0 \le t \le 1-r \\ 0 & ; \quad 1-r \le t \le 1 \end{cases}$

(4.1) becomes

$$\int_0^1\int_0^1 [Z_1(\mu C_1 dU_2) + Z_{1t}(-\mu dU_2) + U_1\left(1+\mu d^2\right)U_2$$
$$+ h_1(\mu C_2 dU_2(x,t+r) + Z_1(\mu C_2 dU_2(x,t-r)))] dtdx$$

or

$$\int_0^1\int_0^1 \left[Z_1(\mu C_1 dU_2 + \mu C_2 dP_2) + Z_{1t}(-\mu dU_2) + U_1\left(1+\mu d^2\right)U_2\right] dxdt$$
$$+ \int_0^1\int_{-r}^0 h_1(\mu C_2 dU_2(x,t+r)) dtdx \qquad (4.2)$$

Set (4.2) to be

$$\int_0^1\int_0^1 \left[Z_1 \bar{Z}_{12} + Z_{1t} \bar{Z}_{12t} + U_1 \bar{Z}_{22}\right] dxdt + \int_0^1\int_{-r}^0 h_1 \bar{Z}_{32} dxdt$$

Also, let

$$\alpha(x,t) = \mu C_1 dU_2(x,t) + \mu C_2 dP_2(x,t) \tag{4.3a}$$

$$\beta(x,t) = -\mu dU_2(x,t) \tag{4.3b}$$

We note that $\left(\beta(x,t) - \bar{Z}_{12}(x,t)\right)$ is a continuous function and it is continuously differentiable over $0 \le t \le 1$.

Thus if

$$\int_0^1\int_0^1 \left[Z_1\left(\alpha - \bar{Z}_{12}\right) + Z_{1t}\left(\beta - \bar{Z}_{12t}\right)\right] dxdt = 0 \tag{4.4}$$

and by involking the consequence of the result in [5] we have as before

$$\frac{\partial}{\partial t}\left[\beta - \bar{Z}_{12t}\right] = \alpha - \bar{Z}_{12} \tag{4.5}$$

which implies that

$$\bar{Z}_{12tt}(x,t) - \bar{Z}_{12}(x,t) = \beta_t - \alpha$$

$$= -\mu dU_{2t}(x,t) - \mu C_1 dU_2(x,t) - \mu C_2 dP_2(x,t)$$

$$= g(x,t) \tag{4.6}$$

In order to again obtain a unique solution, we append the appropriate initial conditions and solve the arising equation in the next section.

ANALYTIC SOLUTION TO THE SECOND ORDER EQUATION PROBLEM (P5):

Solve;

$$\frac{d^2}{dt^2}Z(x,t)-Z(x,t)=g(x,t) \qquad (4.7)$$

Subject to:

$$Z(x,0)=l_0 \quad ; \quad Z_t(x,0)=k_0 \qquad (4.8)$$

where $0 \leq x, t \leq 1$

Taking the Laplace transform of (4.7) in t-space, we have

$$s^2 Z(x,s)-sZ(x,0)-Z_t(x,0)-Z(x,s)=g(x,s) \qquad (4.9)$$

After applying conditions (4.8), we get

$$Z(x,s)=\frac{g(x,s)}{s^2-1}+\frac{s l_0}{s^2-1}+\frac{K_0}{s^2-1} \qquad (4.10)$$

Taking the Laplace transform of (4.8) in t-space, we have

$$Z(x,t)=\int_0^t g(x,s)\text{Sinh}(t-s)ds+l_0\text{Cosh}t+K_0\text{Sinh}t \qquad (4.11)$$

To determine l_0 and K_0 uniquely, we proceed as before;

$$l_0=\alpha(x,0)=\mu C_1 dU_2(x,0)+\mu C_2 dP_2(x,0) \qquad (4.12)$$

$$Z(x,1)=\alpha(x,1)=\mu C_1 dU_2(x,1)+\mu C_2 dP_2(x,1) \qquad (4.13)$$

$$=\int_0^1 g(x,s)\text{Sinh}(1-s)ds+l_0\text{Cosh}1+K_0\text{Sinh}1$$

Using (4.12) in (4.13), we have;

$$K_0=[\mu C_1 dU_2(x,1)+\mu C_2 dP_2(x,1)$$
$$-\int_0^1 g(x,s)\text{Sinh}1(1-s)ds+(\mu C_1 dU_2(x,0)+\mu C_2 dP_2(x,0))\text{Cosh}1]/\text{Sinh}1 \qquad (4.14)$$

substituting value for

$\int_0^1 g(x,s)\text{Sinh}(1-s)ds$ from remark (R1), we get

$$K_0 = [\mu C_1 dU_2(x,1) + \mu C_2 dP_2(x,1) - \{-\beta(x,0)\text{Sinht} + \int_0^t \beta(x,s)\text{Cosh}(t-s)ds$$
$$-\int_0^t \alpha(x,s)\text{Sinh}(t-s)ds\} - \int_0^t \alpha(x,s)\text{Sinh}(t-s)ds$$
$$+(\mu C_1 dU_2(x,0) + \mu C_2 dP_2(x,0))\text{Cosh}1]/\text{Sinh}1$$

Or

$$K_0 = [\mu C_1 dU_2(x,1) + \mu C_2 dP_2(x,1) - \mu dU_2(x,0)\text{Sinht}$$
$$-\int_0^t \mu dU_2(x,s)\text{Cosh}(t-s)ds + \int_0^t(\mu C_1\, dU_2(x,s) + \mu C_2 dP_2(x,s))\text{Sinh}(t-s)ds$$
$$+(\mu C_1\, dU_2(x,0) + \mu C_2\, dP_2(x,0))\text{Cosh}1]/\text{Sinh}1 \qquad (4.15)$$

Lastly, we substitute (4.12) and (4.15) in (4.11) and noting that $P_2(x,t) = U_2(x,t+r)$ we get

$$Z(x,t) = [-\mu dU_2(x,0)\text{Sinht} - \int_0^t \mu dU_2(x,s)\text{Cosh}(t-s)ds$$
$$+(\mu C_1 dU_2(x,s) + \mu C_2\, dU_2(x,t+r)\text{Sinh}(t-s) + (\mu C_1 dU_2(x,0) + \mu C_2 dU_2(x,r))\text{Cosht})$$
$$+\{\mu C_1 dU_2(x,1) + \mu C_2 dU_2(x,1+r) - \mu dU_2(x,0)\text{Sinht}$$
$$-\int_0^t \mu dU_2(x,s)\text{Cosh}(t-s)ds +$$
$$\int_0^t (UC_1 dU_2(x,s) + \mu C_2 dU_2(x,s+r))$$
$$\text{Sinh}(t-s)ds + \mu C_1\, dU_2(x,0) + \mu C_2\, dU_2(x,r)\}\text{Cosh}1]\frac{\text{Sinht}}{\text{Sinh}1} \qquad (4.16)$$

$$\overline{Z}_{22}(x,t) = B_{22}\, Z(x,t) = \mu_1(1+\mu d^2)U_2 \qquad 0 \le x,t \le 1 \qquad (4.17)$$
$$\overline{Z}_{32}(x,t) = B_{32}\, Z(x,t) = \mu C_2 dU_2(x,t+r) \qquad -r \le t \le 0 \qquad (4.18)$$

5. THIRD 2ND ORDER EQUATION

To determine B_{13}, B_{23} and B_{33}, we set $Z_2 \equiv U_2 = 0$. This implies that $Z_{2t} = 0$ and $Z_2(x, t\text{-}r) = h_2(x, t)$

$$\begin{bmatrix} B_{13} & \bar{h}_2 \\ B_{23} & \bar{h}_2 \\ B_{33} & \bar{h}_2 \end{bmatrix} = \begin{bmatrix} \bar{Z}_{13} \\ \bar{Z}_{23} \\ \bar{Z}_{33} \end{bmatrix}$$

$$= \int_{0}^{1}\int_{-r}^{0} [Z_1(\mu C_1 C_2 h_2) + Z_{1t}(-\mu C_2 h_2) + U_1(\mu C_2 dh_2) + h_1(\mu C_2^2 h_2)]dtdx \qquad (5.1$$

$$\int_{0}^{1}\int_{0}^{r} [Z_1(x,s)[\mu C_1 C_2 h_2(x,s) + Z_{1t}(x,s)[-\mu C_2 h_2(x,s)] + U_1(x,s)[\mu C_2 dh_2(x,s)]$$
$$+ h_1(x,s)[\mu C_2^2 h_2(x,s)]]]dtdx$$

$$= \int_{0}^{1}\int_{0}^{r} [Z_1(x,s)[\mu C_1 C_2 h_2(x,s)] + Z_{1t}(x,s)[-\mu C_2 h_2(x,s)]$$
$$+ U_1(x,s)Z_{23}(x,s) + h_1(x,s)Z_{33}(x,s)]]dtdx \qquad (5.2)$$

Using our previous argument we set (5.2) to be

$$\int_{0}^{1}\int_{0}^{r} [Z_1(x,s)\bar{Z}_{13} + Z_{1t}(x,s)\bar{Z}_{13t} + U_1(x,s)\bar{Z}_{23} + h_1(x,s)\bar{Z}_{33}] \qquad (5.3)$$

Alsolet;

$$\alpha(x,s) = \mu C_1 C_2 h_2 \qquad (5.4a)$$

$$\beta(x,s) = \mu C_2 h_2 \qquad (5.4b)$$

We note again that $(\beta(x, s) - \bar{Z}_{13}(x, s))$ is continously differentiable over $0 \le x, t \le 1$ as consequence of result in [5].

Thus if

$$\int_{0}^{1}\int_{0}^{r} [Z_1(x,s)\{\alpha(x,s) - Z_{13}(x,s)\} + Z_{1t}(x,s)\{\beta(x,s) - \bar{Z}_{13t}\}]dtdx = 0 \qquad (5.5)$$

and by using the consequence of the result we have again

$$\frac{d}{dt}[\beta(x,s) - Z_{13t}(x,s)] = \alpha(x,s) \qquad (5.6)$$

which implies that

$$\overline{Z}_{13t}(x,s) = \overline{Z}_{13}(x,s) = \beta_t(x,s) - \alpha(x,s)$$

$$= -\mu C_1C_2h_{2t} - \mu C_1C_2h_2 = P(x,s) \qquad (5.7)$$

To obtain a unique solution we append the initial conditions.

ANALYTIC SOLUTION TO THE THIRD ORDER EQUATION PROBLEM (P6)

Solve

$$\frac{\partial^2}{\partial t^2}Z(x,t) - Z(x,t) = P(x,t) \qquad (5.8)$$

subject to

$$Z(x,0) = \mu_0 \quad ; \quad Z_t(x,t) = q_0 \qquad (5.9)$$

where $0 \le x, t \le 1$

Taking the Laplacetransform of (5.8) in t – space we have after applying condition (5.9)

$$Z(x,s) = \frac{P(x,s)}{S^2-1} + \frac{S\mu_0}{S^2-1} + \frac{q_0}{S^2-1} \qquad (5.10)$$

inverse transform in t-space we have

$$Z(x,s) = \int_0^S P(x,s)\,\mathrm{Sinh}(s-\tau)d\tau + \mu_0 \mathrm{Cosh}s - q_0 \mathrm{Sinh}S \qquad (5.11)$$

$$0 \le S \le r$$

On the basis of our previous arguments, we have

$$\mu_0 = \alpha(x,0) = \mu C_1C_2h_2(x,0) \qquad (5.12)$$

and

$$q_0 = \{\mu C_1C_2h_2(x,r) - \mu C_2Ch_2(x,0)\mathrm{Sinh}r + \int_0^r \mu C_2h_2(x,r)\,\mathrm{Cosh}(r-\tau)d\tau$$

$$+ \int_0^r \mu C_1C_2h_2(x,r)\,\mathrm{Sinh}(r-\tau)d\tau - \mu C_1C_2h_2(x,0)\,\mathrm{Cosh}r\}/\mathrm{Sinh}r \qquad (5.13)$$

$$\int_0^r p(x,\tau)\,\mathrm{Sinh}(r-\tau)d\tau$$

$$= -\beta(x,0)\,\mathrm{Sinh}r + \int_0^r \beta(x,r)\,\mathrm{Cosh}(r-\tau)d\tau \; - \int_0^r \alpha(x,r)\,\mathrm{Sinh}(r-\tau)d\tau$$

$$= \mu C_2h_2(x,0)\,\mathrm{Sinh}r - \int_0^r \mu C_2h_2(x,r)\,\mathrm{Cosh}(r-\tau)d\tau$$

$$= \mu C_2 h_2(x,0)\,\text{Sinhr} - \int_0^r \mu C_2 h_2(x,r)\,\text{Cosh}(r-\tau)\,d\tau$$

$$- \int_0^r \mu C_1 C_2 h_2(x,r)\,\text{Sinh}(r-\tau)\,d\tau \qquad (5.14)$$

Finally, using (5.14), (5.13) and (5.12) in (5.11). we obtain :

$$Z_{13}(x,t) = \mu C_2 h_2(x,0)\text{Sinht} - \int_0^t \mu C_2 h_2(x,s)\,\text{Cosh}(t-s)ds$$

$$- \int_0^t \mu C_1 C_2 h_2(x,s)\text{Sinh}(t-s)ds + \mu C_1 C_2 h_2(x,0)\text{Cosht}$$

6. APPLICATION OF THE ECGM ALGORITHM

The Extended Conjugate Gradient Method (ECGM) is described as follows:

$$+ \frac{\text{Sinht}}{\text{Sinhr}}[\mu C_1 C_2 h_2(x,r) - \mu C_2 h_2(x,0)\text{Sinhr}$$

$$+ \int_0^r \mu C_2 h_2(x,s)\,\text{Cosh}(r-s)ds + \int_0^r \mu C_1 C_2 h_2(x,s)\,\text{Sinh}(r-s)ds$$

$$- \mu C_1 C_2 h_2(x,0)\,\text{Coshr}] \qquad (5.15)$$

$$Z_{23}(x,t) = \mu C_2 dh_2(x,t) \quad ; \quad 0 \le t \le r \qquad (5.16)$$

$$Z_{33}(x,t) = \mu C_2^2 h_2(x,t) \quad ; \quad 0 \le t \le r \qquad (5.17)$$

Z_{13}, Z_{23}, Z_{33} vanishes when r = 0.

Guess the first element of the descent sequence z_0 and compute the remaining members of the sequence using the following algorithm:

$$p_{00} = -g_0 \qquad (6.1a)$$

$$z_{i+1} = z_i + \alpha_i p_i \; ; \quad \alpha_i = \frac{<g_i, g_i>_H}{<p_i B p_i>_H} \qquad (6.1b)$$

$$g_{i+1} = g_i + \alpha_i B p_i \qquad (6.1c)$$

$$p_{i+1} = -g_{i+1} + \beta_i p_i \; ; \quad \beta_i = \frac{<g_{i+1}, g_{i+1}>_H}{<g_i g_i>_H} \qquad (6.1d)$$

where g_i is the gradient of $J(z_i, u_i, \mu)$ and

$$p_i = \begin{bmatrix} p_{z,i} \\ \\ p_{u,i} \end{bmatrix} = \begin{bmatrix} p_z(z_i, u_i, \mu) \\ \\ p_u(z_i, u_i, \mu) \end{bmatrix} \qquad (6.2)$$

Furthermore,

$$J_i = J(z_i, u_i, \mu)$$

$$J_{z,i} = J_z(z_i, u_i, \mu)$$

$$J_{u,i} = J_u(z_i, u_i, \mu) \qquad (6.2b)$$
$$p_{z,i} = p_z(z_i, u_i, \mu)$$
$$p_{u,i} = p_u(z_i, u_i, \mu)$$

with

$$p_z(z_i, u_i, \mu) = \int_0^x \int_0^t J_z(z_i, u_i, \mu) dt dx$$

$$p_u(z_i, u_i, \mu) = \int_0^x \int_0^t J_u(z_i, u_i, \mu) dt dx$$

and $H = w_2^1[0,1] \times L_2[0,1]$ as defined in (2.1) with $z_i = z_i(x,t)$, $u_i = u_i(x,t)$.

From (6.1b) we need to obtain B_{pi} to compute the step length and in turn the descent sequence elements. Now, the step length α_i in the ECGM algorithm (6.1) – (6.2) given by

$$\alpha_i = \frac{< g_i, g_i >_H}{< p_i, Bp_i >_H}$$

Can be computed for different sequence

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