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DELAY DIFFERENTIAL EQUATIONS AND FINITE-DIMENSIONAL ITERATED MAPPING

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ABSTRACT

Delay-differential Equation's (DDE) are currently been used to model high dimensional dynamical systems which exhibit very complex hyper chaotic behaviour.

An attempt has been made to replace the singularly perturbed differential-delay equation which is a continuous infinite dimensional dynamical systems with finite dimensional iterated mappings.

INTRODUCTION

In many Physical situations the causality principle imposes the inclusion of retarded actions. The dynamics is thus modeled by differential-delay equations of the form [1].

$$\dot{v}(t) = F(v(t), v(t-T), \mu), \tag{1}$$

where F is a non linear function, T is the delay time, μ is a control parameter and the state variable v is N - dimensional. Many different models have been introduced in various domains such as optics, biology and physiology. In almost all of the systems, non linear interactions pertain only to the delayed feedback of a scalar variable y,

$$\dot{y}(t) = -\gamma y(t) + f(y(t-T), \mu) \tag{2}$$

There are a number of chronic and acute diseases in which a primary symptom is the altered periodicity of some observable. For example, the irregular breathing patterns in adults with Cheyme-stokes respiration and the fluctuations in peripheral white blood cell counts in chronic granulocytic leukemia (CGL). According to Mackey and Glass [2] previous theoretical studies of the control of respiration and the control of hematopoiesis have associated disease processes with oscillatory instabilities in mathematically complex models. The onset of disease is associated with bifurcations in the dynamics of first order differential-delay (DDE) equations which model physiological systems. This mathematical model (DDE) of physiological systems predict the existence of regimes of periodic and aperiodic dynamics, similar to those encountered in human disease.

Consider the ordinary differential equation

$$\dot{x} = \lambda - \gamma x \tag{3}$$

where x is a variable of interest, t is time, and λ and γ are positive constants giving the production and decay rates, respectively of x. Then x=λ\γ in the limit of t→ ∞. In many physiological systems λ and γ are not constants but depend on the value of x at some earlier time. Thus the instantaneous rate of change of x at a time t will depend on x_T, the value of x at time (t-T)

The Mackey Glass (MG) mathematical model for haematologic disorders given by Mackey and Glass in [2] is in the following form

$$\dot{x} = \frac{ax_T}{1 + x_{T^n}} - bx, \; x_T \equiv x(t-T) \tag{4}$$

Most investigations on the MG system deals with a = 0.2, b = 0.1 and n = 10 The state of the system strongly depends on the delay time T.

Chaotic oscillations at T≥Tc = 16.8 are preceded by a stable steady state and successively doubled periodic orbits.

An electronic oscillator which simulates the Mackey-Glass evolution has been designed by Namajunas et al [3]. The circuit contains a tunable delay unit, a non-linear device and an RC filter. It provides an easy way to generate high dimensional signals.

Chow and Paret [4] showed that the equation that describes an optically bistable device is

$$\dot{x}(t) = -x(t) + \lambda [1 - \sin x(t-1)] \tag{5}$$

while the equation that describes the production of red blood cells is given to be

$$\dot{x}(t) = -x(t) + \lambda x(t-1)^8 e^{-x(t-1)} \tag{6}$$

In the present paper we aim at exploring the dynamical properties of this more general class of dynamical systems (DDE). In order to facilitate the numerical investigation, we wish to restrict ourselves to the discrete time frame by attempting to match discrete mappings with differential delay equations.

We have been motivated by the introduction of coupled map lattices by Kaneko [5] where discretization of both space and time variables still preserved many characteristic features of the spatio-temporal complexity displayed by partial differential equations. This made possible extensive simulations of such phenomena.

THE MODEL AND RESULTS

Given a delay differential equation

$$\dot{x}(t)=cx(t-r) \quad (7)$$

where r is the delay time

We attempt to solve equation(7) using the method of Driver [6] and others [7] by initially specifying a function on some interval of length r, say [to-r, to] and we try to satisfy equation (7) for $t \geq$ to.

Set

$$x(t) = \theta(t) \text{ for } t_o - r \leq t \leq t_0 \quad (8)$$

Where θ is some given function. Then we seek a continuous extension of θ into the future, to a function x which satisfies equation (7) for $t \geq$ to if we take $\theta(t) \equiv \theta_o$, a positive constant and solve equations (7) and (8) on [to, to + r] equation (7) then becomes

$$x(t) = - c\theta_o \text{ with initial condition } x(t_o) = \theta_o$$

The solution is

$$x(t) = c\theta_o (t - t_o) \text{ for } t_o \leq t \leq t_o + r \quad (9)$$

Now that X is known up to (to + r), we consider the interval $[t_o + r, t_o + 2r]$. There equation (7) becomes

$$x(t) = -c\theta_o + c^2\theta_o (t - r - t_o) \quad (10)$$

with initial condition obtained from (9);

$$x(to + r) = \theta_o - cr\theta_o$$

$$x(t) = \theta o - cr\theta_o - c\theta_o(t) + c\theta o(t_o+r) + (c^2\theta_o t^2/2) - c^2\theta (t_o+r)^2/2 - c^2\theta_o(r+t_o)t + c^2\theta o(r+t_o)^2 \quad (11)$$

Consider equation (8)

$$x(t) = x(t_o) = \theta(t) = \theta o \text{ at } to \quad (12)$$

$$x(t_o + r) = \theta_o - c\theta_o (t_o + r - t_o) \text{ at } t_o + r$$

$$= \theta_o - c\theta_o r \text{ at } t_o + r \quad (13)$$

$$x(t_o+2r) = \theta_o - 2r\theta_o + c^2r^2\theta_o/2 \text{ at } t_o+2r \quad (14)$$

equations 12, 13 and 14 represents finite dimensional iterated mappings. Numerically, a delay differential equation of the form

$$\dot{x}(t)=-0.5x(t-1.4) \qquad (15)$$

was simulated.

Fig 1 shows the x(t) for equation 15 and the corresponding finite dimensional iterated map.

DISCUSSION AND CONCLUSION

Our result shows the possibility of matching discrete mappings with differential delay equations, which is a primary concern that is relating a dynamical system in an infinite dimensional phase space to the one-dimensional system.

The dynamic structures such as equilibria, periodic orbits and chaotic attractor found in discrete mappings is also preserved together with stability properties in the differential-delay system. This a fairly complete understanding of differential-delay system does emerge from the knowledge of the discrete mapping.

REFERENCES

1. Hepri S., Giacomelli G, Politi A. and Arecchi F. T [1993] High-demensional chaos in delayed dynamical Systems. Physica D 70 pp 235-249.
2. Mackey M. C and Glass L. (1976): Oscillation and Chaos in Physiological Control Systems, Science 197,pp 287 - 289.
3. Namajunas A., Pyragos K and Tamasevicius A. [1995] An electronic Analog of the mackey-Glass system. Physics Letters A 201 pp. 42-46.
4. Chow S. N and J. Mallet-Paret (1983): Coupled Nonlinear Oscillators, pp 7 -12 North-Holland Publishing Company.
5. Kaneko K (1989) Chaotic but Regular Posi-Nega Switch among coded Attractors by Chuster Size Variations. Phy. Rev. Lett. 63, pp 219-223
6. Driver R. D (1977): Ordinary and Delay Differetial Equations Springer-Verlag New York Inc . pg. 225.
7. Bellman R. and Cooke K.(1963): Differential Equations. Academic Press INC (London) pp 98 - 126.

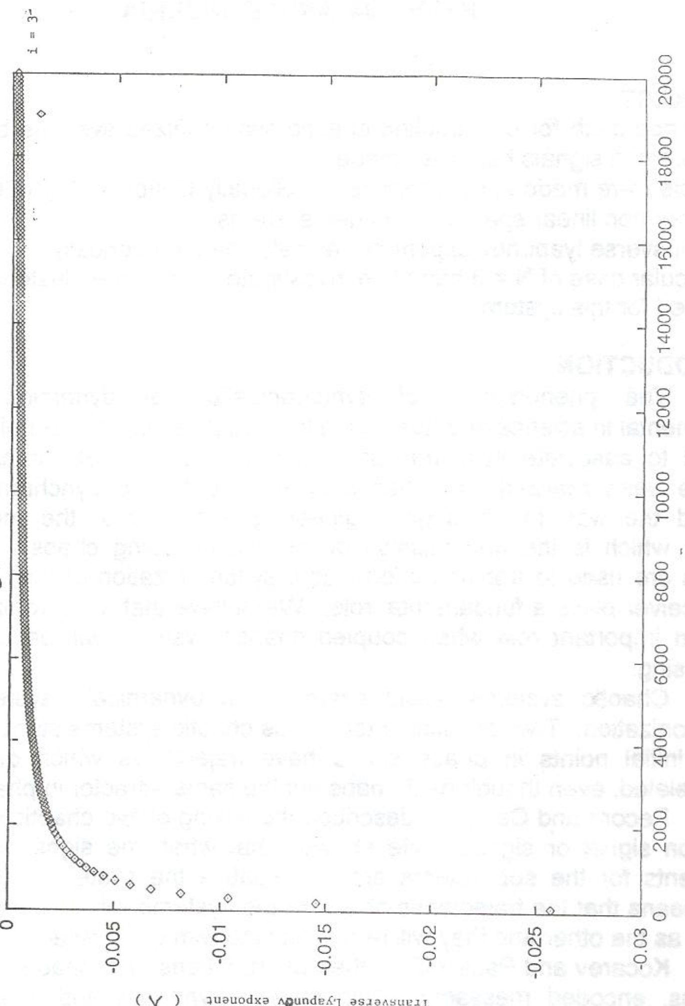

Fig. 3: Graph of λ Versus n

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SYNCHRONIZATION OF CHAOTIC SYSTEMS

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ABSTRACT

A new approach for constructing chaotic synchronized systems by linking them with common signals has been made.

Attempts were made to synchronize the Globally Coupled Map (GCM) which is a model for non linear spatially extended systems.

The transverse lyapunov exponent was calculated numerically.

A particular case of N = 3 has been investigated and some clusterings have been observed for this system

INTRODUCTION

The phenomenon of synchronization of dynamical systems is fundamental in science and has a wealth of applications in technology. While it is natural to associate synchronization with periodic signals, it has in the last decade been realized that chaotic systems can also synchronize [1]. This opened the way to envisage engineering applications, the most developed among which is the transmission of information using chaos. When chaotic signals are used to transmit information, synchronization of the transmitter and the receiver plays a fundamental role. We believe that synchronization will also play an important role when coupled chaotic systems will be used for signal processing.

Chaotic systems would seem to be dynamical systems that defy synchronization. Two identical autonomous chaotic systems started at nearly the same initial points in phase space have trajectories which quickly become uncorrelated, even though each maps out the same attractor in phase space.

Pecora and Carrol [1] described the linking of two chaotic systems with a common signal or signals. He showed that when the signs of the lyapunov exponents for the subsystems are all negative the systems will synchronize. This means that the trajectories of one of the systems will converge to the same values as the other and they will remain in step with each other.

Kocarev and Panlitz [2] in their effort to construct (chaotic) synchronized systems, encoded messages using chaotic dynamics and considered cases where the information signal drives the dynamical system that is used in the transmitter.

The scalar signal which is transmitted from the transmitter to the receiver is a function of the transmitter state variables and the information signal. If the receiver synchronizes with the transmitter, the information signal can be recovered exactly without the reconstruction error that typically occurs with other encoding methods based on synchronization. This modulation technique not only yields a transmission without errors but also a more secured encoding. This new synchronization method is based on the fact that it is possible to consider more general decompositions of a given dynamical system.

$$\dot{Z} = F(Z) \tag{1}$$

Then the decomposition into sub systems proposed by Pecora and Carroll [1].Starting from a chaotic autonomous system, one can always formally rewrite it in different ways as a non- autonomous system

$$\dot{x} = f(x, s(t)) \tag{2}$$

with some driving s(t) = h(x) or s = h (x,s). let

$$\dot{y} = f(y,s(t)) \tag{3}$$

be a copy of the non-autonomous system that is driven by the same signal s(t). If the differential equation for the difference

$$\begin{aligned} e &= x - y \\ e &= f(x,s) - f(y,s) = f(x,s)-f(x-e,s), \end{aligned} \tag{4}$$

possesses a stable fixed point at e = o then there exists for the systems (2) and (3) a synchronized state x=y that is stable. This can be proved using stability analysis of the linearized system for small e or using (global) lyapunov functions.

The stability is checked numerically using the fact that synchronization occurs if all conditional lyapunov exponents of the non autonomous system (2) are negative. The system (2) is a passive system that tends to a fixed point when not driven. Therefore the decomposition given by h and f is called an active- passive decomposition (APD) of the original dynamical system (1). The fact that all conditional lyapunov exponents of (2) are negative does not exclude chaotic solutions.

Synchronization modes have been investigated in ensembles of interacting maps with chaos by Dmitriev et al recently [3]. According to them, synchronization mode stability is determined by two parameters of order: the Lyapunov exponent of the partial dynamics of a separate map, and by a parameter determined form the coupling matrix spectrum Synchromization of chaotic systems have been discussed using the example of two coupled skew tent maps by Hasler and Maistrenko [4]. The skew tent maps have been coupled in two different ways leading to quite different global dynamic behaviour especially when the ideal system is perturbed by parameter mismatch or noise.

This work ;motivated our attempt to synchronize the globally coupled map (GCM) introduced by Kaneko [5]. Globally coupling in dynamical systems yields a host of very novel features and this class of complex systems is of considerable importance in modelling phenomena as diverse as Josephson junction arrays, vortex dynamics in fluids and even evolutionary dynamics, biological information processing and neurodynamics

GCM is a dynamical system of N elements consisting of local mappings as well as an additive average - type interaction term, through which the global information influences the individual elements. The GCM has been synchronized by adding a dissipative coupling to it.

In addition to sychronization, some clusterings were observed for the particular case of N = 3.

THE MODEL

The explicit form of the GCM introduced by Kaneko (5) is the following

$$X_{n+1}(i) = (1-\varepsilon)f(X_n(i)) + \varepsilon/N \sum f(X_n(j)) \qquad (5)$$

where n is a discrete time step and i is the index of the elements (i =1,2......N). The function f(x) is chosen to be the well known dissipative chaotic logistic map.

By adding a dissipative coupling to the GCM in equation (5), the system becomes

$$X_n + 1(i) = (1-\varepsilon)f(X_n(i) + \frac{\varepsilon}{N}\sum_{j=1}^{N} f(X_n(j)) + \sum_{i\neq j}^{N} Kij(X_n(j) - X_n(i) \qquad (6)$$

The term

$$\sum_{i\neq j}^{N} Kij(X_n(j) \_ X_n(i))$$

represents the dissipative coupling where Kij is the coupling strength between elements i and j. At Kij = 0, the system represents the standard GCM. For E= 0.1 and a = 1.99, all elements of the lattice behave chaotically in time ie (all lyapunov exponents are positive and there is no clustering or synchronization in the standard GCM)

NUMERICAL SIMULATION

We have numerically simulated the model given by equation (6).Numerical experiments were carried out with N = 3.By taking a randomly chosen set of initial conditions, the lyapunov exponent which is a measure of the sensitive dependence upon initial conditions and a characteristic of chaotic behaviour according toBaker and Gollub [6] is simulated.

To identify synchronization the deviation Xn(i) - Xn(j) is calculated. This characteristic is finite for unsynchronized state and vanishes for synchronized state. Also the maximal transverse lyapunov exponent λ(k) of the identity synchronization manifold $X_n(i) = X_n(j)$ is calculated using the method of Pyragas [7].

DISCUSSION AND CONCLUSION

Figures 1,2 and 3 show the variation of the maximal transverse lyuapunov exponents with time for the three different elements. Since N = 3 for some Kij $\neq$ 0, Clustering is also observed for N = 3

At Kij = 0, the maximal transverse lyapunov exponent coincides with the maximal lyapunov exponent of the GCM. The synchronization threshold corresponds to the coupling strength at which the transverse lyapunov exponent vanishes.

Our analysis shows that synchronization is possible in GCM models and also that the synchronization condition is completely determined by the maximal lyapunov exponent,

This result is useful for constructing synchronized systems with a possible application to securing communication.

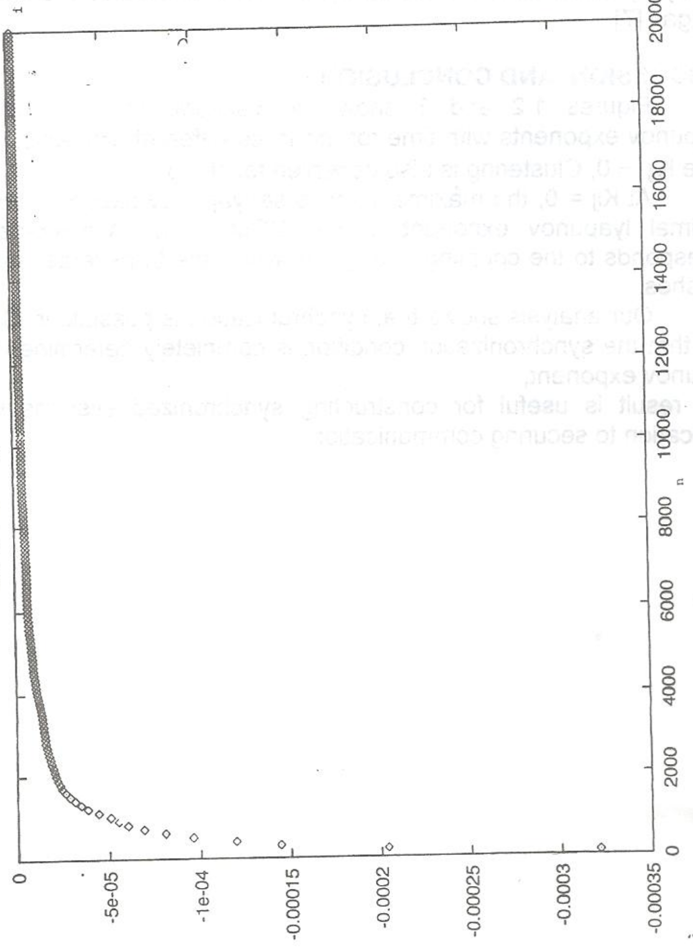

Fig. 1: Graph of λ Versus n

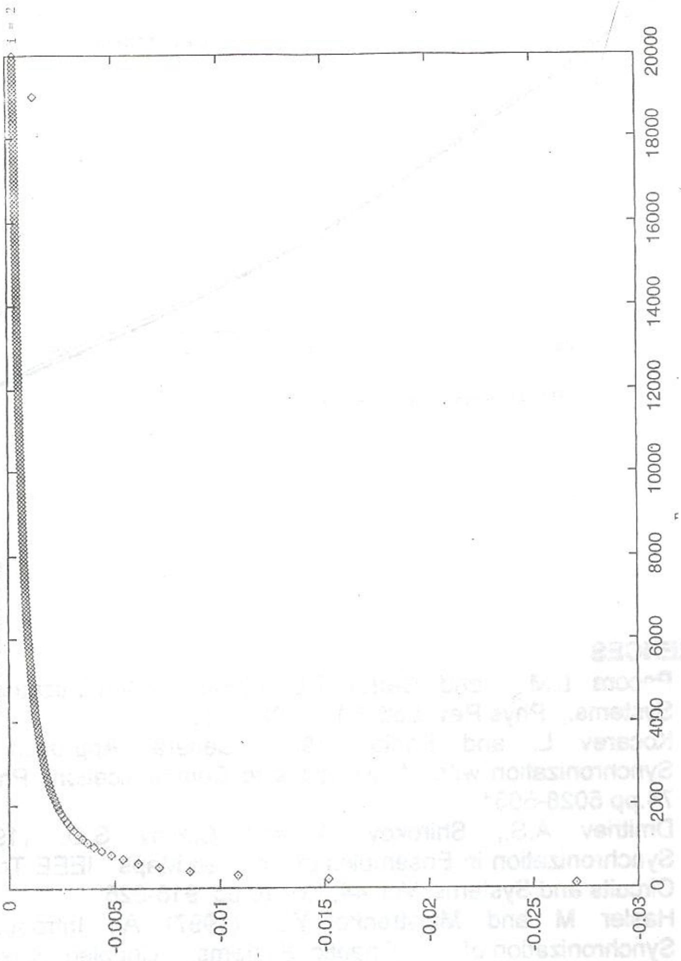

Fig. 2: Graph of λ Versus n

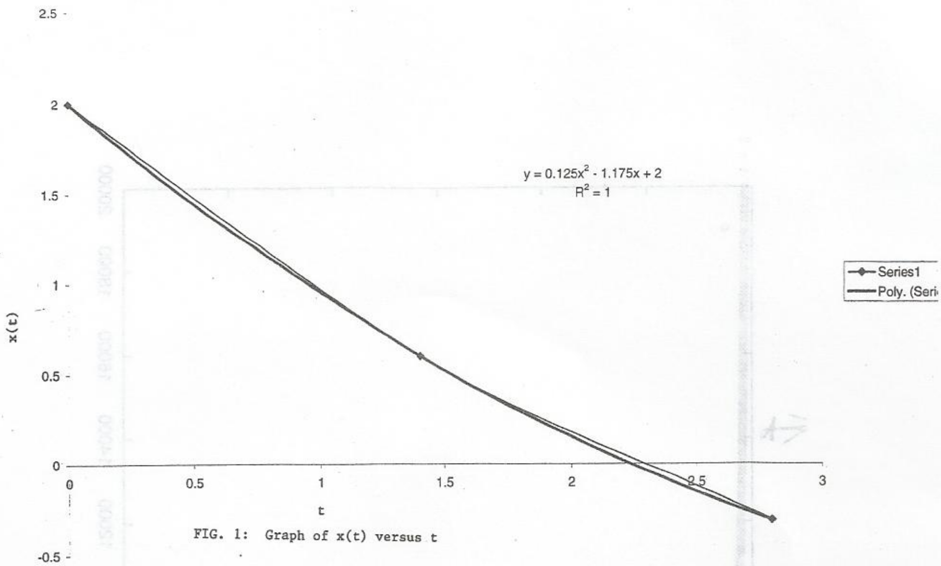

FIG. 1: Graph of x(t) versus t

REFERENCES

1. Pecora L.M. and Carrol T.L. (1990) Synchronization in Chaotic Systems., Phys Rev. Lett. 64 pp. 821-824
2. Kocarev L. and Parlitx (1995) General Approach for Chaotic Synchronization with Applications to Communication Phys. Rev. Lett. 74.pp 5028-5031
3. Dmitriev A.S., Shirokov M and Stakov S.O. (1997) Chaotic Synchronization in Ensembles of Coupled Maps. IEEE Transactions on Circuits and Systems. Vol. 44, No. 10 pp. 918-926
4. Hasler M and Maistrenko Y.L. (1997) An Introduction to the Synchronization of Chaotic Systems: Coupled Skew Tent Maps IEEE Transactions on circuits and systems. Vol 44, No 10, pp 856-866.
5. Kaneko K (1989) Chaotic but Regular Posi-Nega Switch among coded Attractors by Chuster Size Variations. Phy. Rev. Lett. 63, pp 219-223
6. Baker G.L. and Gollub J.P. (1990) Chaotic Dynamics pp 112-113 Published by University of Cambridge.
7. Pyragas K. (1992): Continuous control of Chaos by Self Controlling Feedback. Phys. Lett A 170 pp. 421-428.