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ERROR ESTIMATION IN THE GAUSS AND NEWTON - COTES QUADRATURE SCHEMES FOR WEAK SOLUTIONS OF QUANTUM STOCHASTIC DIFFERENTIAL EQUATION

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ABSTRACT
Gauss and Newton –Cotes quadrature schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations (QSDE) driven by certain operator valued stochastic processes associated with creation, annihilation and gauge operators of quantum field theory in a local convex space are introduced and their error estimates are studied. The work is accomplished within the frame work of Hudson – Parthasarathy formulation of quantum stochastic calculus, Results concerning the consistency and convergence of the schemes in the topology of the locally convex space of solution are presented. This generalised analogous results for classical initial value problems to he equivalent form of noncommutative quantum stochastic differential equations involving unbounded linear operators on a Hilber space.

1. INTRODUCITON

This paper studies the convergence, consistency and error estimates of quadrature schemes of Gauss and Newton-Cotes types for computing weak solutions of Lipschitzian quantum stochastic differential equations. These equations are driven by certain operator valued processes associated with the basic field operators of quantum field theory, introduced by Hudson and Parthasarathy [11] and using the formulations of Ekhaguere [4]. The general theory of quantum stochastic differential equations which are noncommutative generalizations of classical stochastic differential equations have recently undergone rapid developments [2,7,9-11,14,21]. Ekhaguere in his paper [4,5,6] examined analytical properties of solutions of quantum stochastic differential inclusions. These inclusions are of Lipschitzian, hypermaximal monotone and of evolution types. His works generalize some analytical results on quantum stochastic differential equations. There are some other interesting analytical results on quantum stochastic differential equations on* -bialgebras and those in the framework of Ito-Clifford stochastic calculus (see [2,7,21]).

However, there have not been corresponding developments in the numerical solutions of quantum stochastic differential equations and inclusions.

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Consequently, this work is aimed at developing numerical schemes for quantum stochastic differential equations since approximate solutions of these equations play prominent role in applications. Our result is a natural generalization of Gauss and Newton – Cotes schemes for the numerical solutions of Volteral integral equations of the second kind to the equivalent form of quantum stochastic differential equations involving unbounded linear operators on a Hilbert space. Furthermore, our schemes in this framework do not require approximations of driving processes such as generations of Brownian increments during implementations as done in the case of weak and strong Taylor schemes in the classical setting (see [12]).

The rest of the paper is organized as follows. In section 2, we introduce the Gauss and Newton-Cotes quadrature schemes for QSDE and present some important notations and definitions. Our main result on the convergence of these schemes in the space of noncommutative stochastic processes is proved in section 3. Section 4 is devoted to the study of error estimates in the quadrature schemes. In section 5, we discuss quantum formulations of classical Ito equation and some generalizations of the weak Taylor schemes for Ito processes to the present noncommutative quantum setting. We now present some notations and definitions.

For a given pre-Hilbert space D with completion H, Γ(H) denotes the Boson Fock space over H and L^{+}_{w} (D,H) denotes the linear space of all linear operators from D into H with the property that the domain of the operator adjoint contains D. For fixed Hilbert spaces γ and $\Re$, we define $A \equiv L^{+}_{W}(ID\underline{\otimes}IE, \Re \otimes \Gamma(L^2_\gamma(IR_+)))$ where ID is a pre-Hilbert subspace of $\Re$and IE is the subspace of $\Gamma(L^2_\gamma(IR_+))$ generated by the exponential vectors. The completion of A with respect to the topology generated by the family of seminorms $\|x\|_{\eta\varepsilon} = |\langle \eta, x\xi\rangle|$, for $\eta,\xi, \in ID\underline{\otimes}IE, \chi \in A$ is noted by $\widetilde{A}$. Thus for $I \subseteq IR_+$ a map $X : I \to \widetilde{A}$ is called a stochastic process indexed by I. In this work, we adopt the definition of spaces $Ad(\widetilde{A})$, $Ad(\widetilde{A})_{wac,}$ $L^{P}_{10c}(\widetilde{A})$, $L^{\infty}_{\gamma loc}(IR_+)$ and the integrator processes $\Delta_\pi, A_g^{+}, A_f, f, g \in L^{\infty}_{\gamma,loc}(IR_+)$, $\pi \in L^{\infty}_{B(\gamma),loc}(IR_+)$, aş in the references [1,4,5,6].

We now state the first fundamental formula of Hudson and parthasarathy [11] that will be frequently employed in the following analysis.

1.1 Theorem

Let p, q, u, v,∈ $L^2_{loc}(\widetilde{A})$ and let M be their stochastic integral. If η,ξ,∈ $ID\underline{\otimes}IE$ with

$$\eta = c \otimes e(\alpha), \xi = d \otimes e(\beta) c, d \in ID, \alpha, \beta \in L^{\infty}_{\gamma,loc}(IR_+) \text{ and } t \geq 0,, \text{ then}$$

$$<\eta, M(t)\xi>$$
$$= \int_0^t <\eta, \{<\alpha(s), \pi(s)\beta(s)>\gamma\, p(s) + <f(s), \beta(s)>_\gamma\, q(s) + <\alpha(s), g(s)>_\gamma\, u(s)v(s)\}\xi > ds \quad 1.1$$

For E,F, G,H lying in $L^2_{loc}[I \times \widetilde{A}]$, we consider the quantum stochastic differential equation in integral form given by

$$X(t) = X_0 + \int_{t_0}^t (E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s) + G(s, X(s))dA_g^+(s) + H(s, X(s))ds),\ t \in I \quad 1.2$$

Where the integral in equation (1,2) is understood in the sense of Hudson and Parthasarathy [11].

However, Ekhaguere [4] has shown that equation (1.1) is equivalent to the following first order initial value nonclassical ordinary differential equation

$$\frac{d}{dt} <\eta, X(t)\xi> = p(t, X(t))(\eta, \xi)$$
$$X(t_o) = X_0,\ t \in [t_0, T] \quad 1.3$$

Where (t_0, X_0) is some fixed point of $I \times \widetilde{A}$, and for $\eta, \xi, \in ID \underline{\otimes} IE$ with

$\eta = C \underline{\otimes} e(\alpha)$ and $\xi = d \otimes e(\beta)$, define $\mu_{\alpha\beta}, \gamma_\beta, \sigma_\alpha : I \mapsto \mathbb{C}$ by

$$\mu_{\alpha\beta} = <\alpha(t), \beta(t)>_\gamma$$
$$\gamma_\beta(t) = <f(t), \beta(t,)>_\gamma$$
$$\sigma_\alpha(t) = <\alpha(t), g(t)>_\gamma,\ t \in I.$$

To these functions, we associate the maps μE , γF, σG, P from $I \times \widetilde{A}$ into the set of sesquilinear forms on $ID \underline{\otimes} IE$ defined by

$$(\mu E)(t,x)(\eta,\xi) = <\eta, \mu_{\alpha\beta}(t)E(t,x)\xi>$$
$$(\gamma F)(t,x)(\eta,\xi) = <\eta, \gamma_\beta(t)F(t,x)\xi>$$
$$(\sigma G)(t,x)(\eta,\xi) = <\eta, \sigma_\alpha(t)G(t,x)\xi>$$
$$P(t,x)(\eta,\xi) = (\mu E)(t,x)(\eta,\xi) + (\gamma F)(t,x)(\eta,\xi) + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi) \quad 1.4$$

$\eta,\xi \in \mathbb{ID}\underline{\otimes}\mathbb{IE}, (t,x) \in I \times \widetilde{A}$ where $H(t,x)(\eta,\xi) := <\eta, H(t,x)\xi>$

Sometimes we shall have cause to write P in the form

$$P(t,x)(\eta,\xi) = <\eta, P_{\alpha\beta}(t,x)\xi>$$

where

$$P_{\alpha\beta} : I \times \widetilde{A} \to \widetilde{A}$$

is given by

$$P_{\alpha\beta}(t,x) = \mu_{\alpha\beta}(t)E(t,x) + \gamma_\beta(t)F(t,x) + \sigma_\alpha(t)G(t,x) + H(t,x)$$

for $(t,x) \in I \times \widetilde{A}$.

Equation (1.2) is known to have a unique weakly absolutely continuous solution $\Phi : I \to \widetilde{A}$ for the Lipschitzian sesquilinear form valued map P satisfying the Caratheodory conditions. This can be proved by method of successive approximations similar to the procedure adopted in [2] using the first fundamental formula of hudson and Parthasarathy [11 (see [1,4,5]).
In the next section, we shall introduce quadrature formulae of Gauss and Newton – Cotes types for solving equation (1.3). We shall prove that the scheme is convergent in $\widetilde{A}$ and discuss error estimation in section 3.

2. THE GAUSS AND NEWTON – COTES QUADRATURE SCHEMES

We assume the following in what follows, for arbitrary $\eta,\xi \in \mathbb{ID} \otimes \mathbb{IE}$. The map $(t,x) \to P(t,x)(\eta,\xi\ )$ is continuous and Lipschitzian with Lipschitz function $\{K^P_{\eta\xi}(t)$ on $[t_0, T]$ i.e

$$|P(t,x)(\eta,\xi) - P(t,y)(\eta,\xi)| \le K^P_{\eta\xi}(t) + \|x - y\|_{\eta\xi}, \quad x, y \in \widetilde{A}$$

We choose a regular mesh $t_i = t_o + ih$ where $h = \frac{T - t_0}{N}$ is the fixed steplength, for some positive integer N.
Our approach in this work for finding numerical solution to quantum stochastic differential equation

(ii) By equation (2.4) and equation (1.1) of Theorem (1.1) above,

$$< \eta,(X_i - X_0)\xi > = \sum_{j=0}^{i} W_{ij} \int_{t_j}^{t_{j+1}} P(t_j, X_j))\,(\eta,\xi)ds$$

$$= \sum_{j=0}^{i} W_{ij} < \eta, \left(\int_{t_j}^{t_{j+1}} (E(t_j,X_j)dA_\pi(s) + F(t_j,X_j)dA_f(s) + G(t_j,X_j)\,dA_g^+(s) + H(t_j,X_j)ds\right)\xi >$$

by the first fundamental formula of Hudson and Parthasarathy [11] given by equation (1.1). Since η, $\xi \in ID \otimes IE$ are arbitrary, the quadrature scheme (2.4) is equivalent to the discrete scheme

$$X_i - X_0 \sum_{j=0}^{i} W_{ij} \int_{t_j}^{t_{j+1}} (E(t_j,X_j)dA_\pi(s) + F(t_j,X_j))\,dA_f(s) + G(t_j,X_j)\,dA_g^+(s) + H(t_j,X_j)\,ds)$$

for solving equation (1.2) in the space $\widetilde{A}$ of stochastic processes.

(iii) Since the map $(t,x) \rightarrow P(t,\chi)\,(\eta,\xi)$ is complex valued, equation (2.4) can be written in terms of the real and imaginary components of $P(t, X)\,(\eta,\xi)$ and that of $< (\eta, X\xi)$ as follows:

$$\mathrm{Re} < \eta, X_i\xi > = \mathrm{Re} < \eta, X_0\xi > + h \sum_{j=0}^{i} W_{ij}\, \mathrm{Re}P(t_j, X_j)(\eta,\xi) \qquad 2.5$$

and

$$\mathrm{Im} < \eta, X_i\xi > = \mathrm{Im} < \eta, X_0\xi > + h \sum_{j=0}^{i} W_{ij}\, \mathrm{Im}P(t_j, X_j)(\eta,\xi) \qquad 2.6$$

(iv) For each *I*, the set

$$\{W_{ij}, j = 0,1,2,\ldots,i\}$$

represents the weights for an (*i*+1) point quadrature rule of Newton – Cotes type (equally spaced points) for the interval [0,ih]. The repeated trapezoidal rule for the interval [0,ih] has weights W_{ij} given by $W_{i0} = W_{ii} = ½$ $W_{ij} =1, j=1,2\ldots\, i-1$. This is the simplest example of quadrature rule of Newton – Cotes type. In this case, equation (2.4) reduces to

$$<\eta, X_i\xi> = <\eta, X_0\xi> + \frac{h}{2}[P(t_0, X_0)(\eta,\xi) + P(t_i, X_i)(\eta,\xi)]$$
$$+ h\sum_{j=1}^{i-1} P(t_j, X_j)(\eta,\xi) \qquad 2.7$$

(v) By writing the approximation of the integral appearing in equation (2.2) in terms of the real and imaginary parts of the integrand, we have

$$\int_{t_o}^{t_i} \operatorname{Re}P(s,X(s))\ (\eta,\xi)ds = h\sum_{j=0}^{i} W_{ij}\operatorname{Re}P(t_j,X(t_j))\ (\eta,\xi) + E_{t_i}[\operatorname{Re}P] \quad 2.8$$

$$\int_{t_o}^{t_i} \operatorname{Im}P(s,X(s))\ (\eta,\xi)ds = h\sum_{j=0}^{i} W_{ij}\operatorname{Re}P(t_j,X(t_j))\ (\eta,\xi) + E_{t_i}[\operatorname{Im}P] \quad 2.9$$

since the maps $t \to \operatorname{Re}P(t,x(t))(\eta,\xi)$ and $t \to \operatorname{Im}P(t,x(t))(\eta,\xi)$ are continuous and real valued for arbitrary $\eta,\xi \in ID \otimes IE$ for $t \in [0,t_i]$, by classical quadrature existence Theorem (4.1) there exist real numbers $\{W_{ij}, j = 0,1,2...,i\}$ such that equations (2.8) and (2.9) are exact when $\operatorname{Re}P(t,x(t))(\eta,\xi)$ and $\operatorname{Im}P(t,x(t))(\eta,\xi)$ are polynomials of degree less than or equal to i, for each $\eta,\xi \in ID \otimes IE$. Consequently, equation (2.2) is well defined.

2.2 Definition

The method (2.4) is called explicit if $W_{ii} = 0$, $i = 1,2,\ldots,N$ and implicit if $W_{ii} \neq 0$ for at least one value of i.

Thus, for the explicit case, we have

$$<\eta, X(t_o)\xi> = <\eta, X_0\xi>$$
$$<\eta, X_i\,\xi> = <\eta, X_0\xi> + h\sum_{j=1}^{i-1} W_{ij}\,P(t_j, X(t_j))(\eta,\xi) \qquad 2.10$$

and for the implicit case, the system is given by

$$<\eta, X(t_o)\xi> = <\eta, X_0\xi>$$
$$<\eta, X_i\,\xi> = <\eta, X_0\xi> + h\sum_{j=1}^{i} W_{ij}\,P(t_j, X(t_j))(\eta,\xi) \qquad 2.11$$

For the implicit case, the equation defines X_i implicitly and the i^{th} equation must be solved by an iterative technique, which generates a sequence of values $X^{\gamma}_{i,\eta\xi}$, $r = 0,1,2,...$ define by

$$<\eta, X_i^{(r)}\xi> = <\eta, X_0\xi> + h\sum_{j=1}^{i} W_{ij}\, P\left(t_j, X_j\right)(\eta,\xi) + W_{ii} P\left(t_i, X_i^{(r-1)}\right)(\eta,\xi) \qquad 2.12$$

for arbitrary $\eta,\xi \in ID \underline{\otimes} IE$

3. CONVERGENCE AND CONSISTENCY OF QUADRATURE METHODS

3.1 Definition.

(i) The Numerical Method (3.4) is said to be convergent if the solution of the approximating set of equations converges to the solution of the exact problem as steplength h tends to zero, i.e. if

$$\lim_{h \to 0} \left\| X\left(t_o + ih\right) - X_i \right\|_{\eta\xi} = 0 \qquad 3.1$$

with $ih = t - t_0$ fixed.

(ii) The quadrature formula

$$\int_{t_o}^{t_{o+ih}} P(s, X(s))(\eta,\xi)\, ds = h\sum_{j=0}^{i} W_{ij} P\left(t_j . X\left(t_j\right)\right)(\eta,\xi) + E_{t_i}\left(P_{\eta\xi}\right) \qquad 3.2$$

is said to be consistent if

$$\left|E_{t_i}\left[P_{\eta\xi}\right]\right| \to 0$$

as $h \to 0$ with ih fixed.

We shall show that if the quadrature formula (3.2) is consistent then the resulting method is convergent. First we state a lemma due to Mocarsky [15]

3.2 Lemma.

Assume That A,B, P are positive real numbers and { q_i, i = 0, 1.2....,N}is a set of real numbers such that

$$\left|q_i\right| \le A\sum_{j=0}^{i-1}\left|q_j\right| + B \text{ for } i = k, k+1,..., N, \text{ and } \sum_{i=0}^{k-1}\left|q_i\right| \le P \qquad 3.3$$

Then

$$|q_i| \le (B + AP)(1+A)^{i-k}, \text{ for } i = k, k+1, \ldots, N$$

Furthermore, if A = rh and t = ih, then

$$|q_i| \le (B + rhP)\exp(rt) \qquad 3.4$$

3.3 Theorem. Assume that the quadrature formula (3.2) is consistent, then the resulting method (2.4) is convergent.

Proof At $t = t_i$, i = k, k+1,..., N, on substituting (3.2) in (2.1) we have

$$<\eta, X(t_i)\xi> = <\eta, X_0\xi> = +h\sum_{j=0}^{i} W_{ij}P(t_j, X(t_j))(\eta,\xi) + E_{t_i}(P_{\eta\xi})$$

i = k,k+1,..., N and the corresponding approximating equations are

$$<\eta, X_i\xi> = <\eta, X_0\xi> + h\sum_{j=0}^{i} W_{ij}P(t_j, X_j)(\eta,\xi) \quad i = k, k+1, \ldots, N$$

Thus we have

$$<\eta, X(t_i)\xi> - <\eta, X_i\xi> = h\sum_{j=0}^{i} W_{ij}[P(t_j, X_j(t_j))(\eta,\xi) - P(t_j, X_J)(\eta,\xi)] + E_{t_i}[P_{\eta\xi}] \quad i = k,k+1,\ldots,N$$

$$\therefore \ \|X(t_i) -, X_i\|_{\eta\xi} \le h\sum_{j=0}^{i}|W_{ij}|\,\|K^P_{\eta\xi}P(t_j)\|\,\|X(t_j) - X_j\|_{\eta\xi} + |E_{t_i}[P_{\eta\xi}]|$$

$$\le hC_{\eta\xi}\sum_{j=0}^{i}|W_{ij}|\,\|X(t_j) - X_j\|_{\eta\xi} + |E_{t_i}[P_{\eta\xi}]|$$

We let

$$W = \max_{i,j}|W_{ij}|,\ C_{\eta\xi} = \max_{[t_0,T]}|K^P_{\eta\xi}(t)|$$

$$e_i = X(t_i) - X_i,\ e = \max_{0\le i\le k-1}\|e_i\|_{\varepsilon\xi}$$

we have from the last inequality

$$\|e_i\|_{\eta\xi} < \frac{hC_{\eta\xi}W}{1-hC_{\eta\xi}W}\sum_{j=0}^{i-1}\|e_j\|_{\varepsilon\xi} + \frac{1}{1-hC_{\eta\xi}W}\left|E_{t_i}\left[P_{\eta\xi}\right]\right|$$

i=k, k +1 ,…,N

provided that h is small enough such that $1 - hC_{\eta\xi}W \neq 0$ for each η,ξ ID $\otimes$ IE. The last inequality satisfies the conditions of lemma (3.2). In the Lemma, we put

$$B = \frac{1}{1-hC_{\eta\xi}W}\left|E_{t_i}\left[P_{\eta\xi}\right]\right|,$$

$$A = \frac{hC_{\eta\xi}W}{1-hC_{\eta\xi}W}, \quad r = \frac{C_{\eta\xi}W}{1-hC_{\eta\xi}W},$$

$$\sum_{i=0}^{k-1}\|e_i\|_{\eta\xi} \le ke = P$$

By the conclusion of Lemma (3.2), we have

$$\|e_i\|_{\eta\xi} \le \frac{\left|E_{t_i}[P]\right| + hC_{\eta\xi}Wke}{1-hC_{\eta\xi}W}\exp\frac{\left(WC_{\eta\xi}\right)}{1-hC_{\eta\xi}W}ih \qquad 3.5$$

Hence $\|e_i\|_{\eta\xi} \to 0$ as $h \to 0$ with ih fixed

4 ERROR ESTIMATES

We first present a brief summary of techniques and results on classical quadrature rules which are employed later in this section.
We present two different types of quadrature formulas of degree d which are adaptable for the solution of quantum stochastic differential equation (1.3). There are

(i) Newton-Cotes quadrature formulas
(ii) Gauss quadrature formulas

The following theorems, whose proof can be found in Stroud [18], concerns existence of quadrature formulas which are exact for polynomials.

4.1 Theorem .

The $f \in C[t_o, T]$. Then given N distinct points $t_1, t_2, t_3, \ldots t_N \in [t_o, T]$ we can find constants $w_1, w_2, \ldots, w_N$ such that the formula

$$\int_{t_0}^{T} w(s)f(s)ds = \sum_{i=1}^{N} w_i f(t_i) + E[f]$$

is exact, that is $E[f] = 0$ whenever $f \in \Pi_{N-1}[t_o,T]$.

4.2 Theorem .
The N- point repeated quadrature formulas are a sequence of Riemann sums. Therefore as $N \to \infty$, a sequence of such formulas converges to the true value of the integral whenever the integral exists.

4.3 Remark.
It well that if the point t_i , i = 1,2,..., N, are zeros of orthogonal polynomial of degree N corresponding to the weight function w on intervals $[t_o,T]$, the Gauss formula

$$\int_{t_0}^{T} w(t)f(t)dt = \sum_{i=1}^{N} w_i f(t_i) + E[f] \text{ has degree } 2N-1$$

The points $t_1, t_2, \ldots, t_N$ in a Gauss formula are the zeros of orthogonal polynomial P_N of degree N corresponding to the weight function on the interval $[t_o,T]$. Existence and uniqueness of Gauss formulas and their properties are established by the next theorem.

4.4 Theorem. [18]
(i) Let $t_1, t_2, \ldots, t_N$ denote the zeros of the N^{tth} degree orthogonal polynomial P_N for $[t_o,T]$. with weight function w. Assume that the weight $w_1, w_2, \ldots w_N$ are found such that the formula

$$\int_{t_0}^{T} w(t)f(t)dt = \sum_{i=1}^{N} w_i f(t_i) + E[f] \qquad (4.1)$$

is exact for all polynomial of degree $\leq N - 1$, then the formula has degree $2N - 1$.
(ii) If a formula

$$\int_{t_0}^{T} w(t)f(t)dt = \sum_{i=1}^{N} w_i f(t_i) + E[f] \qquad (4.2)$$

is exact for all polynomials of degrees $\leq 2N-1$ then the points in this formula must be zeros of the orthogonal polynomial P_N for the interval $[t_o,T]$ and weight function w.
(iii) In a Gauss formula (4.1), all the w_i , i = 1,2,... N are positive and satisfy

$$w_i = \frac{1}{R_{N-1}P'_{N-1}(t_k)P_{N-1}(t_k)} \tag{4.3}$$

(iv) If $f \in C[t_o, T]$, the sequence of Gauss formulas

$$\sum_{i=1}^{N} w_{i,}Nf(t_{i,N})$$

Satisfies

$$\lim_{N\to\infty} \sum_{i=1}^{N} w_{i,}Nf(t_{i,N}) = \int_{t_0}^{T} w(t)f(t)dt.$$

4.5 Remark.

(1) Practical and convenient methods for computing the numbers t_i , w_i in a Gauss formula have been given by Golub and Welsch [8]. It is shown that given the three term recurrence relation for the orthogonal polynomials generated by the weight function, the quadrature rule may be generated by computing the eigenvalues and the first component of the orthonormalized eigenvectors of a symmetric tridiagonal matrix.

(ii) Table of special orthogonal polynomials together with their respective Gauss quadrature rules are available in the literature ([18], [19], [22]). These tables of values may be used to generate quadrature rules for any intervals and weight functions by an appropriate transformation of special intervals to any other interval of interest. We summarize the procedure as follows:

4.6 Theorem [18,22].

(i) Under the linear transformation

$$t = au + b$$

the interval $[t'_o, T']$ of t is converted to interval $[t'_o, T']$ of u where

$$u = \frac{1}{a}(t - b), \quad a = \frac{T - t_o}{T' - t'_0}, \quad b = \frac{t_0T' - Tt'_0}{T' - t'_0}$$

and the integral

$$\int_{t_0}^{T} w(t)f(t)dt = \int_{t'_0}^{T'} w'(u)g(u)du$$

where

$$g(u) = f(au+b),\ w'(u) = w(au+b), \qquad 4.5$$

(ii) The quadrature formula

$$\int_{t_0}^{T} w(t)f(t)dt = \sum_{i=1}^{N} w_i f(t_i) + E[f] \qquad 4.6$$

transforms into

$$\int_{t_0'}^{T'} w(u)g(u)du = \sum_{i=1}^{N} w_i' g(u_i) + E'(g) \qquad 4.7$$

where

$$u_i = \frac{1}{a}(t_i - b),\ w_i' = \frac{1}{a} w_i' \quad i = 1,2,\ldots,N$$

and

$$E'(g) = \frac{1}{a} E(f)$$

(iii) If formula (4.6) has degree d, then (4.7) also has degree d. So the degree of the qudrature formula is invariant under a linear transformation (4.4).

4.7 Remark

(i) The proof of the last theorem can be found in [18], [22].

(ii) The transformation $t = 2u - 1$ converts the interval [-1, 1] into [0,1].

The Gauss – Lagendre rule $\{w_i, t_i\}$ has been tabulated in the literature for $w(t) = 1$ and for the interval [-1,1]

We employ Theorem (4.6) to compute the Gauss – Legendre rules for [0,1] by writing

$$w_i' = \frac{1}{2} w_i,\ u_i = \frac{1}{2}(t_i + 1)$$

Wher t_i, = 1,2,..,N are zeros of Legendre polynomial of degree N which are listed for various values of N in the literature ([18], [19]).

Next we examine error estimates for quadrature rules.

4.8 Definition.

Let $[t_o, T]$ be a finite interval and let $f \in C[t_0, T]$. The error $E(f)$ in an N – point quadrature rule for f is defined by

$$E(f) = \int_{t_0}^{T} w(t)f(t)\,dt - \sum_{i=1}^{N} w_i f(t_i) \qquad (4.8)$$

4.9 Notation (i)

For $S \in IR$, we introduce the notation

$$(S)_+^m = S^m, S \geq 0$$
$$= 0, S < 0$$

(ii) If $f \in C^i[t_0, T]$, then $= M^{(i)} = \sup_{t_0 \leq s \leq T} \left| f^{(i)}(s) \right|$

(ii) $$K_r = \int_{t_0}^{T} \left| E_s \left[(s - t)_+^r \right] \right| dt \qquad 4.9$$

where E_s indicates the error operator defined by equation (4.8) in integrating the function $g(s,t) = (s-t)_+^r$ over s for fixed t..

$$E_s \left[(s-t)_+^r \right] = \int_{t_0}^{T} (s - t)_+^r \, ds - \sum_{i=1}^{N} w_i (t_i - t) +$$

We remark that K_r given by equation (4.9) is called the Peano kernel. Next, we state the Peano's theorem for an N – point quadrature rule.

4.10 Theorem [18,22]

Let $f \in C^q[t_0, T]$. such that $f^{(q+1)}$ is piecewise continuous on $[t_0, T]$. Suppose that an N – point quadrature rule has degree d. Then if r = min (d,q), the quadrature error $E(f)$ satisfies

$$| E(f) | \leq \frac{M^{(r+1)}}{r!} K_r \qquad 4.10$$

4.11 Remark

We can also use inequality (4.10) to bound the error for an N – point Gauss rule. For N-point Gauss – Legendre rule we have

$$|E(f)| \leq \frac{M^{(2N)}}{(2N-1)!}\int_{-1}^{1}\left|E_s(s-t)_+^{2N-1}\right|dt$$

$$\leq \frac{M^{(2N)}}{(2N-1)!}2^{N+2} \cong 4M^{(2N)}\left(\frac{N}{\pi}\right)^{\frac{1}{2}}\left(\frac{e}{N}\right)^{2N} \qquad 4.11$$

The proof of the last theorem can be found in Stroud [18], Young and Gregory [22]

4.12 Remark.
We now discuss application of Peano theorem for error estimates for repeated (M-panel) rules . We let Q_M be an M – panel repeated N – point rule based on a quadrature rule of degree d and order r. Let f satisfy the hypothesis of the Peano's theorem. Then applying the Peano's theorem to each panel, we have from (4.10)

$$|E_M(f)| = |(1-Q_M)f| \leq \frac{M^{(r+1)}}{r!}\sum_{K=1}^{M}\int_{t_0+(k-1)h}^{t_0+kh}\left|E_s\left[(s-t)_+^r\right]\right|dt$$

where

$$h = \frac{T-t_0}{M}.$$

and

$$E_S\left[(S-t)_+^r\right] = \int_{t_0+(k-1)h}^{t_0+kh}(s-t)_+^r\,ds - \sum_{i=1}^{M}w_i\left(t_0+(k-1)h+h\theta_i-t\right)_+^r \qquad 4.12$$

where θ_i are distinct and such that $0 \leq \theta_i \leq 1$, $i = 1,2,\ldots N$.
Now for all s,t appearing in equation (4.12), satisfying $|s-t| \leq h$ and for all y such that $(y)_+ \leq |y|$, we have in general for some A dependent on the rule but not M,

$$\left|E_S\left[(S-t)_+^r\right]\right| \leq Ah^{r+1}$$

and hence

$$|E_M f| \leq MA\frac{M^{(r+1)}}{r!}h^{r+2} = \frac{AM^{(r+1)}}{r!}(T-t_o)h^{r+1} \qquad 4.13$$

We observe that equation (4.13) is valid for repeated Gauss or repeated Newton –Cotes rules. It shows that as h tends to zero, the error for a high order r = min (d,q)-rule reduces more rapidly than for low order rule provided that the integrand f is sufficiently smooth so that r = d. If the integrand is not smooth, then by (4.13), two quadrature rules of degrees $d_1 > q$ and $d_2 > q$ are predicted by (4.13) to converge at the same rate; thus we might as well use the rule of higher degree since we incur no penalty.
By using (4.13), the errors in the repeated Trapezoidal and Simpson's rules over $[t_0,T]$ have been shown to satisfy the following: (cf [3,18]) For repeated Trapezoidal rule,

$$|E(f)| \leq M^{(2)}(T-t_0)\frac{h^2}{12}$$

and for repeated Simpson's rule

$$|E(f)| \leq \frac{M^{(4)}(T-t_0)}{180}h^4 \qquad 4.14$$

where

$$M^{(i)} = \sup_{s\in[t_0,T]}\left|f^{(i)}(s)\right|$$

4.13 Remark
(Applications To Quantum Stochastic Differential Equations)
(i) The quadrature formula (3.2) is equivalent to equations (2.8) and (2.9) of Remark (2.1).

If we employ a sequence of i + 1 – point repeated quadrature rule, then by Theorem (4.2),

$$E_{t_1}\left[\mathrm{Re}\,P_{\eta\xi}\right] \to 0$$

and

$$E_{t_1}\left[\mathrm{Im}\,P_{\eta\xi}\right] \to 0$$

as $h \to 0$ with $i\,h$ fixed. By Theorem (4.4) (iv) we have convergence also for a sequence of Gauss rule. Consequently, the quadrature formula (3.2) is consistent and therefore converges by Theorem (3.3).
(ii) The rate of convergence of the quadrature formula (3.2) can be estimated as follows.

Suppose that for each η,ξ∈ ID ⊗ IE, the map $P_{\eta\xi}(.) := P(.,X(.))(\eta\xi)$ is of class $C^q[t_0,T]$ and $P_{\eta\xi}^{(q+1)}$ is piecewise continuous on $[t_0,T]$. Then the real and imaginary parts of $P_{\eta\xi}(.)$ are also of class $C^q[t_0,T]$ and their derivatives of order q+1 are piecewise continuous on $[t_0,T]$. By linearity of the error functional E_{t_i}

$$\left|E_{t_i}\left[P_{\eta\xi}\right]\right|^2 = \left|E_{t_i}\left[\mathrm{Re}\,P_{\eta\xi}\right]\right|^2 + \left|E_{t_i}\left[\mathrm{Im}\,P_{\eta\xi}\right]\right|^2 \qquad 4.15$$

By theorem (4.10) and equation (4.13) there exist constants A>0, $M_{i,1,\eta,\xi}^{(r+1)}$, $M_{i,2,\eta,\xi}^{(r+1)}$, r =min(d, q) such that

$$\left|E_{t_i}\left[P_{\eta\xi}\right]\right|^2 \le \left[\frac{AM_{i,1,\eta,\xi}^{(r+1)}}{r!}(t_i-t_o)h^{r+1}\right]^2 + \left[\frac{AM_{i,1,\eta,\xi}^{(r+1)}}{r!}(t_i-t_o)h^{r+1}\right]^2$$

$$\therefore \quad \left|E_{t_i}\left[P_{\eta\xi}\right]\right| \le \frac{A\sqrt{2}}{r!}M_{i,\eta,\xi}^{(r+1)}(t_i-t_o)h^{r+1} \qquad 4.16$$

where by Theorem (4.10)

$$M_{i,1,\eta,\xi}^{(j)} = \sup_{t_0\le s\le t_i}\left|\mathrm{Re}\,p_{\eta\xi}^{(j)}(s)\right|$$

and

$$M_{i,2,\eta,\xi}^{(j)} = \sup_{t_0\le s\le t_i}\left|\mathrm{Im}\,p_{\eta\xi}^{(j)}(s)\right|$$

$$M_{i,\eta,\xi}^{(r+1)} = \max\left\{M_{i,1,\eta,\xi}^{(r+1)}, M_{i,2,\eta,\xi}^{(r+1)}\right\}$$

We notice that by inequalities (4.16) and (3.5), the rate of convergence of the quadrature formula (3.2) is faster for a quadrature rule of high order r and slower for a rule of low order. However if the integrand is not differentialble i.e q = 0, then r = 0, so that the rate of convergence of the quadrature error to zero is linear. The foregoing statements imply that for high degree quadrature rules such as Gauss rules, the quadrature converges faster than the N – point repeated Newton –Cotes rules and provided that the map t → P(t,X(t)) (ηξ) is sufficiently smooth. In particular, if we apply the bounds for repeated Trapezoidal and Simpson's rules given by inequality (4.14), we have from (4.16), for the trapezoidal rule,

$$\left| E_{t_i}\left[P_{\eta,\xi}\right]\right| \le \frac{\sqrt{2}}{12} M^{(2)}_{i,\eta,\xi}(t_i - t_o)h^2$$

and for the repeated Simpson rule

$$\left| E_{t_i}\left[P_{\eta,\xi}\right]\right| \le \frac{\sqrt{2}}{180} M^{(4)}_{i,\eta,\xi}(t_i - t_o)h^4$$

This indicates that trapezoidal and Simpson quadrature rules are of order one and three respectively

5. APPLICATION TO CLASSICAL ITO STOCHASTIC DIFFERENTIAL EQUATIONS IN QUANTUM FORM.

In this section as in [11], we take $\Upsilon=\Re=\mathbb{C}$ and write

$$A(t) = A_f(t),\ A_g^+(t) = A^+(t),\ f(t) \equiv g(t) \equiv 1\ \forall t \in \Re_+.$$

Consequently, the Fock space $\Re \otimes \Gamma\left(L^2_\gamma(\Re_+)\right) \equiv \Gamma\left(L^2_\gamma(\Re_+)\right)$. It is well known [cf [9,16,17]) that the creation and annihilation operators $A^+(t)$ and $A(t)$ satisfy the commutation relations $[A(s), A(t)] = [A^+(s), A^+(t)] = 0$ and $[A(s), A^+(t)] = \min\{s,t\}$. 1 on the dense subspace *IE* of $\Gamma\left(L^2_\gamma(\Re_+)\right)$ spanned by the exponential vectors and that the self-adjoint operator –valued adapted process Q given by $Q(t) = A(t) + A^+(t)$ is a Brownian motion. Then the Fock space $\Gamma\left(L^2_\gamma(\Re_+)\right)$ can be realized as $\Gamma^2(\Omega F, W)$ where (Ω, F, W) is the Wiener space (cf.[9,16,17]). In this realization, the vacuum e(0) becomes the function identically 1 on the Wiener space of continuous trajectories and Q(t) becomes the operator of multiplication by the canonical Brownian motion. We identify each random variable x with the operator of multiplication by x; then

$$(Q(t)F)(\omega) = X(t)(\omega)F(\omega) = \omega(t)F(\omega),\ \text{for } F \in L^2(\Omega, F, W)$$

so that

$$Q(t) = A(t) + A^+(t) = \omega(t) \qquad 5.1$$

where w(t) is the evaluation of the Brownian path w at time t. As an element of $L^2(\Omega, F, W)$, the exponential vector is given by

$$e(\alpha)(\omega) = \exp\left(\int_0^\infty \alpha(s) d\omega(s) - \frac{1}{2} \int_0^\infty \alpha^2(s) ds \right) \qquad 5.2$$

where $\int_0^\infty \alpha^2(s)dw(s)$ denotes the Wiener integral and α is a purely imaginary function in $L^2_{\alpha}(IR_+)$, (see [9]). Next we present a result which concerns the quantum analogue of the classical Ito integral for Brownian motion. In what follows E is the expected value function.

5.1 Theorem.
Let F and H be nonanticipating Brownian functionals and E be expected value function such that

$$\int_0^t E\left[f(s,.)^2\right] ds < \infty \text{ and } \int_0^t E\left[H(s,.)^2\right] ds < \infty$$

Then as multiplication operator valued processes

F = (F(t,.);, t ≥ 0) and H = (H(t,.) : t ≥ 0) belong to $L^2_{loc}(\tilde{A})$

Proof. The proof of the above theorem is a simple application of Proposition 5.1 in Hudson and Parthasarathy [11]. The details are contained in [1]

5.2 Remark.
(i) If we put

$$m(t,w) = \int_0^t F(s,w)dw(s) + \int_0^t H(s,w)ds$$

where the first term is the usual Ito-Doob Mean Square integral of F, then following theorem (5.1), and on account of equation (5.1) we can write

$$m(t,.) = \int_0^t (F(s,:)dA(s) + F(s,.)dA^+(s) + H(s,.)\, ds \qquad 5.3$$

(ii) Following equation (5.3), the classical Ito stochastic differential equation in integral form, has the quantum analogue given by

$$X(t,.) = X_0(.) + \int_0^t (F(s,X(S,.))dA(s) + F(s,X(s,.))dA^+(s) + H(s,X(s,.))\,ds) \qquad 5.4$$

(iii) For $\eta,\xi \in IE$ such than $\eta = e(\alpha), \xi = e(\beta)$ where $\alpha,\beta : IR_+ \to \mathbb{C}$ and are bounded functions in $L^2_{\mathbb{C}}(IR_+)$ having purely imaginary values, we employ the function $\mu_{\alpha\beta,}\gamma_\beta$ and σ_α introduced in section 1, to obtain

$$\mu_{\alpha\beta,}(t) = 0$$
$$\gamma_\beta(t) = <1, \beta(t)>_{\mathbb{C}} = \beta(t)$$
$$\sigma_\alpha(t) = <\alpha(t), 1>_{\mathbb{C}} = \bar{\alpha}(t)$$

and putting

$$z(\omega) = \exp\left\{ \int_0^\infty -\alpha(s) + \beta(s))dw(s) - \frac{1}{2}\int_0^\infty (\alpha^2(s) + \beta^2 s)ds \right\} \qquad 5.5$$

we get

$$\mu E(t,X)(\eta,\xi) = 0$$
$$\gamma E(t,X)(\eta,\xi) = <\eta, \gamma_\beta(t)F(t,.X)\xi> = <\eta, \beta(t)F(t,X)\xi>_{\Gamma(L^2{\mathbb{C}}(IR_+))\equiv L^2(\Omega,w)}$$
$$= F[\beta(t)F(t,X)e(\alpha)e(\beta)] = E[\beta(t)F(t,X(t))z(w)]$$
$$\sigma F(t,X(t))(\eta,\xi) = <\eta, \sigma_\alpha(t,X(t))\xi> = <e(\alpha), \bar{\alpha}(t)F(t,X(t))e(\beta)>$$
$$= E[\bar{\alpha}(t)F(t,X(t))z(w)]$$
$$H(t,X(t))(\eta,\xi) = <\eta, H(t,X(t))\xi> = E[H(t,X(t))z(w)]$$

Therefore

$$P(t,X(t))(\eta,\xi) = \mu E(t,X(t))(\eta,\xi) + \gamma F(t,X(t))(\eta,\xi) + \sigma F(t,X(t))(\eta,\xi)$$
$$+ H(t,X(t))(\eta,\xi)$$
$$= E[\beta(t)F(t,X(t))z(w)] + E[\bar{\alpha}(t)F(t,X(t))\,z(w)]$$
$$+ E[H(t,X(t))z(\omega)]$$
$$<\eta,, X(t)\xi> = <e(\alpha), X(t)e(\beta)> = E[X(t,w)z(w)]$$

Thus we have the equivalent form of equation (5.4) given by

$$\frac{d}{dt}E(X(t,w)z(w)) = E[\beta(t)F(t,X(t))z(w)] + E[\bar{\alpha}(t)F(t,X(t))z(w)] + E[H(t,X(t))z(\omega)]$$

$$E(X(t_o)z(w)) = E(X_o z(w)), \text{ Almost all } t \in [t_0, T] \tag{5.6}$$

The quadrature methods which we develop in this work generalise several weak Taylor approximations of Ito stochastic differential equations due to Kloeden and Platen [12] with respect to weak convergence criteria.
By Remark (2.3) the quadrature scheme (2.4) when applied to Ito stochastic differential equation (5.4) in quantum form is equivalent to the scheme

$$X_i = X_0 + \sum_{j=0}^{i} W_{ij} \int_{t_j}^{t_{j+1}} \left(F(t_j, X_j)dA(s) + F(t_j, X_j)dA^+(s) + H(t_j, X_j)ds\right)$$

In the classical setting, this has the form

$$X_i = X_0 + \sum_{j=0}^{i} W_{ij}\left[\int_{t_j}^{t_{j+1}} H(t_j, X_j)ds + \int_{t_j}^{t_{j+1}} F(t_j, X_j)d\,W(s)\right]$$

$$= X_0 + \sum_{j=0}^{i} W_{ij}\left[H(t_j, X_j)h + F(t_j, X_j)\Delta W_j\right] \tag{5.7}$$

Where $\Delta W_j = W(t_{j+1}) - W(t_j), \{W(t), t \geq 0\}$ is the classical Brownian motion . The discrete scheme (5.7 represents several weak Taylor schemes in [12]. By Theorem (3.2), the quadrature scheme (2.4) conver ges in $\widetilde{A}$ with order of convergence equals equals to 1.0 i.e. there exists constants $C_{\eta\xi} > 0 \ni$

$$\|X(t_i) - X_i\|_{\eta\xi} \leq C_{\eta\xi}h$$

i.e

$$|E(X(t_i) - X_i)z(w)| \leq C_{\eta\xi}h \tag{5.8}$$

so that if we choose $\eta = \xi$, $\eta = e(\alpha), \xi = e(\beta)$ where α, β are functions in $L^2_\alpha(IR_+)$ having purely imaginary values such that $\alpha(t) = id(t)$, $\beta(t) = il(t)$, then by equation (5.5)

$$z(w) = \exp\left(\int_0^T d^2(t)\,dt \right)$$

Thus z(w) is nonrandom and hence inequality (5.8) becomes

$$| E(X(t_n) - X_n)| \leq C\partial, C = \frac{1}{z(w)} C_{\eta\xi}, n = 0,1,2,\ldots,N$$

The convergence of our quadrature scheme, in this case, is equivalent to the weak convergence of the appropriate weak Taylor scheme in [12] since in that case the sequence of approximations satisfies

$$| E(X(T)) - E(XN)| \leq Ch$$

For some constant C independent of h and T = Nh.
We note that for g: R → R such that g is of class C^2, the scheme (2.4) can be applied to generate approximations of expectation E(g(X(t))) of functionals of Ito process X(t,w) by employing the classical Ito formula

$$g(X(t,w)) = g(X(t_0)) + \int_{t_0}^{t} [H(s,X(s))\,g'(X(s))$$

$$+ \frac{1}{2} F^2(s,X(s))\,g''(X(s))\,ds + \int_{t_0}^{t} [F(s,X(s))\,g'(X(s))\,dW(s)$$

in quantum form .

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