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DE-SITTER GROUP APPROACH TO THE THEORY OF GRAVITATION

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ABSTRACT

By a suitable choice of metric, curved space-time is realised as a 4-dimensional metric space of class 1, i.e., it is imbedded in (flat) (4+1) de-Sitter space. By this stratagem an alternative derivation is achieved of one of the well-known results of the theory of general relativity, namely, the precession of the perihelia of the planets Mercury, Venus, the Earth, and Icarus. The results are in good agreement with experiment. The derivation leads to the emergence of a *fundamental unit of length*, $a = c^4/(8\pi G\Delta) = 0.481856\text{x}10^{15}\,\text{m}$, where $\Delta = 10^{28}\ \text{J m}^{-2}$ is a *fundamental stress-energy surface density*, and G is the Newtonian universal gravitational constant.

1. Introduction

There is enormous experimental verification of the basic tenets of quantum mechanics on the one hand and of Einstein's special and general theories of relativity on the other hand. Yet it has not been easy to marry these two aspects of physical theory. It has been our view (Maduemezia, 1987) that a theory of relativity intermediate between the special and general theories should do the trick. In particular, the possibilities offered by the natural chain of groups beginning from the conformal group SO(4,2) and ending with the rotation group, SO(3), namely,

$$\text{SO}(4,2) \to \text{SO}(4,1)\square\text{T}(5) \to \text{SO}(4,1) \to \text{SO}(3,1)\square\text{T}(4) \to \\ \to \text{SO}(3,1) \to \text{SO}(3)\square\text{T}(3) \to \text{SO}(3) \quad ;$$

where '□' denotes a semi-direct product and $T(n)$ is a translation group on n-dimensions, has not been fully exploited in physics. In this chain of groups, each semi-direct product is obtained from the non-compact semi-simple group to the left of it by means of group contraction (Segal 1951, Inönü & Wigner 1953, Saletan, 1961)

In particular, the Poincaré group $\text{SO}(3,1)\square\text{T}(4)$, the symmetry group of special relativity, is obtained from the de-Sitter group, *SO*(4,1) by group contraction. An intermediate theory based on *SO*(4,1) should reproduce all the results of the only the *essentially general relativistic effects* in the universe, if these can be identified, to be accounted for by the general theory.

If V_n is an n-dimensional metric space, it can in general be imbedded in a flat space of $n(n+1)/2$ dimensions or less. If the lowest order of flat space in which it can be imbedded is $n + p$, we say that V_n is of class p. It is well known (Eisenhart, 1926) that space-time V_n outside a symmetric mass M, and equipped with the Schwarzschild metric:

$$g_{\mu\nu} = \begin{pmatrix} Z & & & \\ & -1/Z & & \\ & & -r^2 & \\ & & & -r^2 \sin^2\theta \end{pmatrix}; \quad Z = 1 - \frac{2GM}{c^2 r},$$

with respect to the spherical coordinate ordering (t, r, θ, φ), may be imbedded in (flat) 4+2 de-Sitter space with a quadratic form:

$$ds^2 = (dz^1)^2 + (dz^2)^2 - (dz^3)^2 - (dz^4)^2 - (dz^5)^2 - (dz^6)^2.$$

This form is invariant under the Conformal group SO(4,2). The construction for this imbedding is as follows:

$$\alpha = \frac{2GM}{c^2}; \quad Z = 1 - \frac{\alpha}{r}$$

$$z^1 = \sqrt{Z}\cos t;\ z^2 = \sqrt{Z}\sin t;\ z^3 = f(r);$$

$$z^4 = r\sin\theta\cos\varphi;\ z^5 = r\sin\theta\sin\varphi;\ z^6 = r\cos\theta,$$

where $f(r)$ satisfies the equation:

$$\left(\frac{df}{dr}\right)^2 = \frac{1}{Z}\left(\frac{\alpha^2}{4r^4} + \frac{\alpha}{r}\right).$$

To this extent, the static solution of Schwarzschild has as symmetry group the conformal group SO(4,2), and not the group of general linear transformations of space-time. All the results obtainable from the Schwarzschild metric are therefore obtainable from the operations in 4+2 de-Sitter space, S_6, in which space-time is immersed.

The fundamental hyperquadrics (Eisenhart, 1926, section 61) of the flat space S_6 are the two spaces of constant Riemannian curvature,

$$\sum_{\mu=1}^{6} c_\mu \left(z^\mu\right)^2 = \pm R^2 ,$$

where R is an arbitrary constant, and $c_1 = c_2 = c_3 = c_4 = 1 = -c_5 = -c_6$

The group SO(4.2) acts on these two *homogeneous spaces* as a group of motions. The Lie algebra of SO(4,2) is thereby easily realised as an algebra of differential operators (Maduemezia,1967). Since this Lie algebra contains all the dynamical variables of quantum mechanics (momentum, angular momentum, Lorentz boosts, and analogues of the position operator), we can hope to quantize gravity through this channel of correspondences. For now, however, our interest is in finding a metric for V_n under which it is a space of class 1, that is, a metric that would enable us immerse it in one or the other of the two 5-dimensional de-Sitter spaces, (3+2) or (4+1). For definiteness, we choose the latter.

In a (4+1) de-Sitter world, the (invariant) quadratic differential form, or line element, ds^2 , is given (Maduemezia, ,1987 , Eqn. (7.1)) by:

$$ds^2 = -(cd\tau)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (cdt)^2 + a^2 (dg)^2 , \tag{1}$$

where (**x**,t) = (*x,y,z,ct*) is space-time, and *g* is a dimensionless variable *depending on the Newtonian gravitational field* at (**x**,t), and *a* is a fundamental scale of length.

It is the purpose of this paper to show that through the use of this metric, we can derive some of the well-known results of the general theory of relativity that have been confirmed by experiment. We shall in particular look at one issue, namely, the precession of the perihelia of four planets.

2. PRECESSION OF THE PERIHELIA OF THE PLANETS

Let us consider the orbital motion of any one of the planets-*Mercury*, *Venus*, The *Earth* and *Icarus*. We take the x-axis normal to the elliptical orbit, which is in the *x-y* plane. Thus in the spherical coordinate system, (r, θ, φ), the coordinate θ is constant at the value $\pi/2$. Hence the quadratic form (1) becomes

$$ds^2 = -(cd\tau)^2 = (dr)^2 + r^2 (d\varphi)^2 - (cdt)^2 + a^2 (dg)^2 .$$

The (4+1) de-Sitter group acts on the flat 5-dimensional pseudo-Euclidean space with co-ordinates, $(x_1, x_2, x_3, x_4, x_5) \equiv (x, y, z, ag, ct)$

and metric $g_{\mu\nu} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$,

c and *a* being physical constants, *c*, the speed of light *in vacuo*, and *a* a fundamental length to *be determined*.

Now just in same way that we can draw a circle (a one-dimensional curved space) on a flat sheet of paper (2-dimensional flat space), we can imbed *4-dimensional curved space-time* in this *flat 5-dimensional de-Sitter space*. The number of ways in which this can be effected is infinite, but presumably a limited class of such imbeddings is predicated by the physics of the problem on hand.

For the present problem, which is the theory of gravitation, we shall imbed space-time into de-Sitter space by means of the *Ansatz*:

$$g = \sqrt{\left(1 - \frac{2GM_s}{c^2 r}\right)} = \sqrt{\left(1 - \frac{\alpha}{r}\right)} \tag{2}$$

where *G* is the Newtonian constant of gravitation, and M_s is the mass of the sun, so that

$$\alpha = \frac{2GM_s}{c^2} = 2.953364 \text{ x } 10^3 \text{m}$$

Now put $Z = 1 - \alpha/r$. Thus

$$(dg)^2 = \frac{\alpha^2}{4Zr^4}(dr)^2 \tag{3}$$

Hence

$$c^2 = c(r,\theta,\varphi)^2 = c^2\left(\frac{dt}{d\tau}\right)^2 - \left(1 + a^2\frac{\alpha^2}{4Zr^4}\right)\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\varphi}{d\tau}\right)^2 \tag{4}$$

Now the *Lagrangian* $L(r, \theta, \varphi)$ = Total Energy (E) – $mc(r, \theta,\varphi)^2 + GM_s m/r$, where *m* is the mass of the planet. Thus *L* does not contain the variables φ and *t* explicitly. They give rise to 2 conservation laws:

$$\frac{dt}{d\tau} = A, \tag{5}$$

where A is a constant related to the energy;

and

$$r^2 \frac{d\varphi}{d\tau} = J, \quad \text{also a constant.} \tag{6}$$

Now put $r = 1/u$, then after a little algebra, we obtain

$$Zc^2 = Zc^2 A^2 - \left(Z + a^2 \beta u^4\right) J^2 \left(\frac{du}{d\varphi}\right)^2 - J u^2 Z \tag{7}$$

where

$$\beta = \alpha^2/4 \tag{8}$$

Now let

$$T = c^2\left(A^2 - 1\right)/J^2 \tag{9}$$

The constant A^2 is required to be less than 1, i.e., $T < 0$; this corresponds to the fact that the energy of the planet has an upper bound, since it is in a closed orbital state.

Then (7) may be written

$$T(1 - \alpha u) = \left(1 - \alpha u + a^2 \beta u^4\right)\left(\frac{du}{d\varphi}\right)^2 + u^2(1 - \alpha u) \tag{10}$$

Differentiating this with respect to φ, and cancelling out the resulting common factor $(du/d\varphi)^2$, we obtain

$$\begin{aligned} &\frac{d^2u}{d\varphi^2} + u + \frac{\alpha T}{2} = \gamma \\ &= \left(\alpha u - a^2 \beta u^4\right)\frac{d^2u}{d\varphi^2} + \left(\frac{\alpha}{2} - 2a^2 \beta u^3\right)\left(\frac{du}{d\varphi}\right)^2 + \frac{3}{2}\alpha u^2 \end{aligned} \tag{11}$$

Now let $l = -\frac{2}{\alpha T}$. We seek a solution of Eqn. (11) of the form

$$u = u_o + \varepsilon,$$

where

$$u_0 = \frac{1 + e\cos\varphi}{l} = \lambda + \omega\cos\varphi, \text{ say.} \tag{12}$$

e is the *eccentricity* of the planetary orbit, and l is the *semilatus rectum*. The term ε is therefore a correction term, and is expected to be small. Thus in

$$\gamma = \frac{d^2\varepsilon}{d\varphi^2} + \varepsilon , \tag{13}$$

we may put $u = u_o$. We thus obtain

$$\frac{d^2 u_0}{d\varphi^2} = -\omega \cos\varphi; \quad \left(\frac{du_0}{d\varphi}\right)^2 = \omega^2 \sin^2\varphi , \tag{14}$$

and

$$\gamma \approx -\omega\cos\varphi\left(\alpha\, u_0 - a^2\beta\, u_0^4\right) + \left(\frac{\alpha}{2} - 2a^2\beta\, u_0^3\right)\omega^2\sin^2\varphi + \frac{3}{2}\alpha\, u_0^2 \tag{15}$$

Now for the planet Mercury the *semilatus rectum* $l = 5.546 \times 10^{10}$ m .
On the other hand, the *eccentricity* e is 0.2056.
Thus the quantity $\omega = \left(\frac{e}{l}\right) = 3.707 \times 10^{-12}$, which is a small number; and so is λ $= 1/l$. Thus in eqn. (15), we may ignore terms with factors of order $\lambda^2\omega^3$, or lower, namely, $\lambda^2\omega^3, \lambda\omega^4, \omega^5$. With this approximation, Eqn. (15) reduces to

$$\begin{aligned}\gamma = -\omega\cos\varphi\left\{\alpha\lambda + \alpha\omega\cos\varphi - a^2\beta\left(\lambda^4 + 4\lambda^3\omega\cos\varphi\right)\right\} + \\ + \omega^2\sin^2\varphi\left(\frac{\alpha}{2} - 2a^2\beta\lambda^3\right) + \frac{3}{2}\alpha\left(\lambda^2 + 4\lambda\omega\cos\varphi + \omega^2\cos^2\varphi\right)\end{aligned} \tag{16}$$

Now let

$$S = \frac{a^2\alpha}{l^3} = a^2\alpha\,\lambda^3 \tag{17}$$

Then

$$\gamma = \cos\varphi\left[2\alpha\omega\lambda + a^2\beta\lambda^4\omega\right] + \cos^2\varphi\left[6a^2\beta\lambda^3\omega^2\right] + \\ + \left[\frac{\alpha\omega^2}{2} - 2a^2\beta\lambda^3\omega^2 + \frac{3}{2}\alpha\lambda^2\right] \\ = \frac{\alpha}{2l^2}\left[\cos\varphi(4+S/2)e + 3Se^2\cos^2\varphi + (1-S)e^2 + 3\right] \tag{18}$$

Then

$$\frac{d^2u}{d\varphi^2} + u + \frac{\alpha T}{2} = \frac{\alpha}{2l^2}\left(Q + P\cos\varphi + R\cos^2\varphi\right) \tag{19}$$

where

$$P = (4 + S/2)e; \\ Q = 3 + (1-S)e^2; \\ R = 3Se^2, \tag{20}$$

has a solution of the form:

$$u = \frac{1 + e\cos\varphi}{l} + \varepsilon . \tag{21}$$

A particular solution of (19) is obtained by choosing

$$\varepsilon = C + D\cos^2\varphi + F\varphi\sin\varphi \tag{22}$$

Then

$$D = -\frac{\alpha}{6l^2}R; \\ F = \frac{\alpha}{4l^2}P; \\ C = \frac{\alpha}{l^2}\left(\frac{1}{2}Q + \frac{1}{3}R\right), \tag{23}$$

and we obtain the solution

$$u = \frac{1 + e\cos\varphi}{l} + \frac{\alpha}{L^2}\left\{\left(\frac{1}{2}Q + \frac{1}{3}R\right) - \frac{1}{6}\cos^2\varphi + \frac{1}{4}P\varphi\sin\varphi\right\} \tag{24}$$

Of the three terms in {...}, the last is the most important because it is linear in φ, and so grows in time, while the constant term and the term in $\cos^2\varphi$ have upper bounds. We may ignore these last two terms in comparison with the first, and write

$$\begin{aligned} u &= \frac{1 + e\cos\varphi}{l} + \frac{\alpha}{4L^2}P\varphi\sin\varphi \\ &= \frac{1 + e\cos\varphi}{l} + \frac{\alpha}{4l^2}(4 + S/2)e\varphi\sin\varphi \end{aligned} \tag{25}$$

$$S = \frac{a^2\alpha}{l^3} = a^2\alpha\lambda^3 \tag{26}$$

Now, for the planet Mercury, $l = 5.546 \times 10^{10}\ m$; $\alpha = 2.953364 \times 10^3\ m$.

Thus $\frac{\alpha}{4l} = 0.133E-07 \quad << \quad 1$

and

$$\eta = \frac{\alpha}{4l}(4 + S/2) \quad << \quad 1\ , \tag{27}$$

provided that S is of the order of 1, which it is, as we shall see later.
so that

$$\begin{aligned} u &\approx \frac{1 + e[\cos\varphi + \eta\varphi\sin\varphi]}{l} \\ &= \frac{1 + e\cos(\varphi - \eta\varphi)}{l}, \end{aligned} \tag{28}$$

where we have used

$$\cos(\varphi - \eta\varphi) = \cos\varphi\cos(\eta\varphi) + \sin\varphi\sin(\eta\varphi) \approx \cos\varphi + \eta\varphi\sin\varphi\ .$$

Thus at the end of 1 revolution, the phase angle $(1-\eta)\varphi = 2\pi$

$$\varphi = 2\pi + \eta\varphi \tag{29}$$

i.e.,

$$\Delta\varphi = 2\pi\eta = \frac{\alpha\pi}{2l}(4+S/4)\ rad/rev.$$
$$= \frac{\alpha\pi}{2l}(4+S/4)(2.0626481E+05)\ arcsecs/rev. \qquad (30)$$

Now noting that Mercury makes 415.2 revolutions per century, and that experimentally, the precession, $\Delta\varphi$ = *43.11 arcsecs* per century, we have

$$\frac{43.11}{415.2} = \frac{\alpha\pi(4+S/2)2.0626481E+05}{2l}, \qquad (31)$$

which gives

$$S = 4.03562214, \qquad (32)$$

which is consistent with the inequality (27).
Using (32) in (26), we have

$$a = 0.482799 \times 10^{15}\ m \qquad (33)$$

Table 1. Useful constants *
Newton's Gravitational constant, $G = 6.670 \pm 0.006 \times 10^{-11}\ Nm^2kg^{-2}$
Velocity of light $c = 2.997925 \times 10^8$ m/s
Mass of the Sun, $M_s = 1.989769 \times 10^{30}$ kg
1 radian = 2.0626481×10^5 arcseconds

Planet	Semi-major axis A (10^6 km)	No. of orbital revolutions per century	Semi Latus Rectum $l = A(1-e^2)$ *km*	Eccentricity of orbit, *e*
Mercury	57.91	415.2	55.46×10^6	0.2056
Venus	108.21	162.5**	108.20×10^6	0.0068
Earth	149.60	100	149.56×10^6	0.0167
Icarus	161.0	89	50.89×10^6	0.827

(* Weinberg, 1972; ** Kaye & Laby, 1971)

We now use this value of *a* to compute the precessions of the perihelia of the other three planets per century using (30) and the data in Table 1. The results are given in Table 2

Table 2: Precession of the perihelia of the planets ($a = 0.482799 \times 10^{15}$ m)

Planet	No. of orbital revolutions per century	Semilatus rectum l	Experimental value of the angle of precession (secs/century)	This calculation of the angle of precession (secs/century)
Mercury	415.2	5.546E+10	43.11 ± 0.45	43.11
Venus	162.5	10.820E+10	8.4 ± 4.8	6.14
Earth	100	14.956E+10	5.0 ± 1.2	2.62
Icarus	89	5.089E+10	9.8 ± 0.8	11.01

It can be seen that the agreement with experiment is quite good; given the usual reservations over the true definition of $\Delta\varphi$ (Weinberg 1972)

The entity $a = 0.482799 \times 10^{15}$ *m* is a fundamental unit of length in a de-Sitter world, much in the same way that *c* is a fundamental unit of speed. We shall see in the next section that it can be written in terms of known physical constants.

3. LINK WITH GENERAL RELATIVITY

Einstein's fundamental *Ansatz* in the General Theory of Relativity may be written (Kenyon, 1990):

$$\text{Curvature Tensor} = \frac{8\pi G}{c^4}\text{(Stress - Energy Tensor)}$$

Dimensionally speaking, the stress-energy tensor $T^{\alpha\beta}$ is *energy per unit volume* ($J.m^{-3}$). Thus since $c^4/(8\pi G)$ has dimensions $J.m^{-1}$, the curvature tensor has dimension m^{-2}, and relates to the *two-dimensional curved* surface "enclosing the region inside which the stress-energy tensor $T^{\alpha\beta}$ is contained" (Thorne, 1980). We may therefore re-write the entity $c^4/(8\pi G)$ as $a.\Delta$, where a is a fundamental unit of length, and Δ is a fundamental free surface stress-energy (surface Tension). This is guided by the fact that the entity $c^4/(8\pi G)$ has the value 0.481856×10^{43} $J.m^{-1}$ (cf. $a = 0.482799 \times 10^{15}$ m). Thus if we take $\Delta = 10^{28}$ $J.m^{-2}$, then the fundamental constant a in (33) becomes

$$a \approx c^4/(8\pi G\Delta) \quad \textit{meters}$$

Using this value of a to re-compute the precessions of the perihelia of the four planets, we have the results in Table 3, which is practically the same as Table 2.

Table 3. Precession of the perihelia using $a \approx c^4/(8\pi G\Delta)$ *meters*

Planet	Experiment (secs/century)	This theory (secs/century)
Mercury	43.11 ± 0.45	43.05
Venus	8.4 ± 4.8	6.14
Earth	5.0 ± 1.2	2.62
Icarus	9.8 ± 0.8	11.00

The agreement with experiment is again very good.

4. CONCLUSION

We have shown that at least one of the standard results of general relativity truly belongs to a lower member of the hierarchy of relativity theories intermediate between the special and general theories (Maduemezia, 1987). It would be interesting to find out how many more of the standard results of general relativity may be so realised, and how many remain essentially general relativistic. This matter is being investigated.

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