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INTEGRAL INVARIANT OF HAMILTONIAN SYSTEMS REVISITED

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ABSTRACT

In a bounded closed Hamiltonian system of n degrees of freedom with only one constant of motion, one may consider the n^{th} momentum P_n as a 'Hamiltonian' K and the n^{th} coordinate q_n as 'time' τ. Then it is the case that the Lagrangian M corresponding to K is such that the associated action is an invariant. The place of this invariant, particularly in chaotic system, is discussed.

1 INTRODUCTION

Hamiltonian systems constitute a tiny but crucially important subset of dynamical systems, and span the gamut of fundamental (space - time) physics. The power of Hamilton's (canonical) equations of dynamics is the freedom of choosing what are coordinate $\{q_j\}$ and what are conjugate momenta $\{P_j\}$. The Hamiltonian H: $(p,q) \rightarrow \mathbb{R}$ and time t are also conjugates. But the beauty of it is the fact that any of the so - called momenta, say P_n, can infact be called a 'Hamiltonian K, $\equiv -p_n$,with its conjugate coordinate q_n taken as 'time' τ. By considering the equations that govern quantum - mechanical state ψ (q,t),

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \quad ; \quad -i\hbar\frac{\partial\psi}{\partial q_j} = p_j\psi,$$

One easily note (and it can be shown rigorously within classical mechanics - see Akin - Ojo, 1992) that $(-P_n, q_n)$ stand in the same stead as (H,t). (The minus sign in $K \equiv - p_n$ is crucial). Corresponding to the 'Hamiltonian' K is the 'Lagrangian' M (got by Legendre transformation). While in Schroedinger's (differential equation)quantum mechanics the Hamiltonian is the special operator, in Feynman's (path-integral) quantum mechanics the Lagrangian is the special function. In general, the action

$$R = \int L dt$$

of the Lagrangian L is the phase we need in the subsequent "stationary phase approximation" method (Gutzwiller, 1990). It is this need that calls for the Lagrangian M corresponding to the Hamiltonian K. So, to obtain M is the main task of this article. And we shall see that M is an interesting expression, and very useful especially in non - integrable system that portend chaos.

2 ANALYSIS

We consider a closed bounded system of n degrees of freedom with usual Hamiltonian H (p,q), $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$,

$$H : (p,q) \to R \; ; \; H = \sum_{1}^{n} P_j^2 / 2m + V(q_1, q_{2,} \cdots, q_n) \quad (1)$$

$n \geq 2$, $|p|$, $|q| < \infty$; and because system is closed, H(p,q) = constant E, the only constant of motion. Let us call $p_n \equiv -K$ $q_n \equiv \tau$; that is

$$K = -[2m(E - V(q_1, \ldots, q_{n-1}, \tau)) - \sum_{1}^{n-1} p_j^2]^{1/2} \quad (2)$$

with

$$2m V(q_{1,}\ldots q_{n-1,}\tau) + \sum_{j}^{n-1} p_j^2 \leq 2mE$$

The dynamics in 2(n - 1) - dimensional phase space of the system are given by

$dq_j/d\tau = \partial K/\partial p_j$, $dp_j/d\tau = -\partial K/\partial q_j$, j = 1,2,...,n - 1.

Note that K depends on τ, and therefore K is not a constant of motion. That is, K is completely non - integrable.

Since, as one may easily show that, $\partial^2 K/\partial p_j \partial p_k$ is a positive semi - definite (n-1) x (n-1) symmetric matrix, K is a convex function in p. Therefore its Legendre transform M (or convex conjugate) is convex in $u \equiv \partial K/\partial p \in R^{n-1}$. Define

$$u_r \equiv \partial K / \partial p_r = p_r / (\gamma - \sum_{1}^{n-1} P_j^2)^{1/2} \quad (3)$$

where $\gamma = 2m(E-V) = \sum_1^n p_j^2$. The Lagrangian

$$M(u,q) \equiv \sum_1^{n-1} p_j \partial K / \partial p_j - K = \sum_1^{n-1} p_j u_j - K$$

A careful analysis gives

$$\underline{P}^2 = (I + W)^{-1} \underline{U}^2 \gamma^{1/2}, \tag{4}$$

where

$$\underline{P^2} = (p_1^2, p_2^2, \ldots p_{n-1}^2)^+$$

$$\underline{U^2} = (u_1^2, u_2^2, \ldots u_{n-1}^2)^+$$

$$\underline{W} = \begin{pmatrix} u_1^2 & u_1^2 \ldots u_1^2 \\ u_2^2 & u_2^2 \ldots u_2^2 \\ u_{n-1}^2 & u_{n-1}^2 \ldots u_{n-1}^2 \end{pmatrix}$$

and I is the (n-1) x (n-1) unit matrix.
Consequently, we obtain

$$p_r^2 = \gamma u_r^2 / (1 + \sum_1^{n-1} u_j^2) \tag{5}$$

And with $u_n = dq_n/d\tau = 1$, we have

$$M = \gamma^{\frac{1}{2}} [1 + \sum_1^{n-1} u_j^2]^{\frac{1}{2}} = \gamma^{\frac{1}{2}} [\sum_1^n (dq_j / d\tau)^2]^{\frac{1}{2}} \tag{6}$$

Let us compute the action of M

$$S \equiv \int M d\tau = \int \gamma^{\frac{1}{2}} [\sum_1^n m (dq_j / d\tau)^2]^{\frac{1}{2}} d\tau / m$$

$$= \int \gamma^{\frac{1}{2}} [\sum_1^n m (dq_j / dt)^2 (dt / d\tau^2]^{\frac{1}{2}} (d\tau / dt) dt / m$$

$$= \int \gamma^{\frac{1}{2}} [\sum_1^n p_j^2]^{\frac{1}{2}} dt / m \tag{7}$$

But $\gamma = \sum_1^n p_j^2$ Hence

$$S = \int (\sum_{j=1}^{n} p_j^2) dt / m = \int \sum_{1}^{n} p_j (dq_j / dt) dt = \int (\sum_{1}^{n} p_j dq_j) \qquad (8)$$

which is the so called abbreviated action S(q, E), known to be one of the famous integral invariants of Poincare, under all canonical transformations (Golstein, 1950).

3 DISCUSSION

(a) The above may be generalised to Hamiltonian $H(p,q) = K(p) + V(q)$ when K is convex, such as $K(p) \equiv \mu^{ij} p_i q_j$ with μ a positive-definite n x n symmetric matrix. In this case , one must make some transformation to get H into the form given in eq. (1).

(b) Consider the case of periodic system, of period T. The time average of S is

$$\bar{S} = \frac{1}{T} \int_{sT}^{(s+1)T} (\sum_j p_j^2) dt$$

For instance, for the harmonic oscillator,

$$\bar{S} = \oint p dq = ET$$

so that for k circuits, $S_k = kTE$. Then the relevant Green's function is

$$G(q,E) = \sum_{k=0}^{\infty} \exp[i(kTE/\hbar - k\pi)] \qquad (9)$$

$$= [1 - \exp(i(TE/\hbar - \pi))]^{-1}$$

The additional term $k\pi = 2k(\pi/2)$ comes from the turning points of an oscillator (see Gutzwiller 1990). In this case G has poles at $TE - \pi\hbar = 2\pi s\hbar$, i.e. with $T = 2\pi/\omega$

$$E_s = 2\pi(s + \frac{1}{2}) \hbar(2\pi/\omega) = (s + \frac{1}{2})\hbar\omega \qquad (10)$$

which is recognised as the oscillator energy quantization. In this spirit , the integral invariant S may be useful in the quantization of chaotic systems.

(c) We consider the famous non-integrable Hamiltonian

$$H(p,q) \equiv (p_1^2 + p_2^2 + q_1^2 + q_2^2)/2 + q_1^2 q_2 - q_1^2/3 = E$$

analysed by Henon and Heiles (1964) and revisited by Akin Ojo (1994). Let the trajectory return to the Poincare section ($p_1 > 0$, $q_1 = a \neq 0$) at times $\{T_s\}$, and define

$$\bar{S} = \int_{T_s}^{T_{s+1}} (p_1^2 + p_2^2)dt/(T_{s+1} - T_s) = E, \quad s = 1,2,... \qquad (11)$$

We find, computationally, that the time average of S is approximately constant, E, even though system is periodic. The same result is obtained for the quartic Hamiltonian

$$H \equiv (p_1^2 + p_2^2)/2 + q_1^4 + q_2^4 + 2q_1^2 q_2^2)/4 = E$$

(d) In conclusion letting $\wedge_r$ be the time of r^{th} return of trajectory to the Poincare section, and letting

$$S_r \equiv \int_0^{\wedge_r} (\text{kinetic energy})dt$$

we have the Green's function (not the propagator!)

$$G(q,E) \equiv \sum_{r=0}^{\infty} \exp(iS_r/\hbar - i\mu\pi/2)$$

$$\equiv \sum_{r=0}^{\infty} \exp(i \wedge_r \bar{S}_r / \hbar - i\mu\pi/2)$$

where μ depends on turning points (Gutzwiller 1990). This leads to the quantum mechanics of any (even chaotic) Hamiltonian systems.

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