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ONE DIMENSIONAL VISCOELASTIC MODEL OF ARTERIAL BLOOD FLOW

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ABSTRACT

Theoretical and experimental evidences are available showing that the arterial walls are viscoelastic. Hence we propose a model of blood flow in a viscoelastic tube where the pressure depends on the strain and the time-rate of strain. The hyperbolic nature of the constitutive equations was destroyed by this introduction leading to a parabolic equation of the general Burger form. This viscoelasticity introduces a retarding effect in the equations depicting an interplay between the non-linear steeping and the diffusion of a wave. Analytic solution of the vector Burger equation for cylindrical coordinate system in three dimension was obtained.

INTRODUCTION

One of the important steps in understanding the response and state of health of arteries is knowledge of their mechanical properties. For example, a monitoring of mechanical properties would lead to the detection of such pathological states as calcification of the arterial wall. However such monitoring would have to be accomplished in a non- – traumatic way to be reliable. One of the methods adopted to do this is the wave propagation technique Anliker et al (1971). A theoretical study of wave propagation of blood in arteries is used to determine the physical, geometrical and mechanical properties of the vessels and the effects of these parameters on the propagation of velocity and pressure waveforms. Most of these mathematical theories have been based on an unsteady one-dimensional model in which the internal pressure and fluid velocity are averaged over the cross section of the artery. (Anliker et al 1971, Hoogstraten and Smit 1978, Vender Werff 1974, Akinrelere and Ayeni, 1983). In all these arterial blood flow models, the assumption is made that the arterial walls are purely elastic.

MATHEMATICAL ANALYSIS

In this work, we adopt a quasi linear one dimensional model for an incompressible fluid in a distensible tube, based on the assumptions that (i) the wave length is long compared with the tube diameter and (ii) that the tube is constrained from longitudinal motion, and (iii) the wall of the material is assumed to be viscoelastic. Under these conditions, the governing equations are the equations of the continuity of mass and the conservation of momentum.

These are:

$$\frac{\partial A}{\partial t}+\frac{\partial (Au)}{\partial x}+\Psi=0 \qquad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\gamma u\Psi}{A}=f \qquad (2)$$

where A is the Area of the tube, u the average longitudinal velocity along the axial coordinate x, P the pressure, ρ the fluid density and f the frictional term. Ψ is the outflow function along the tube and γ (a constant ≥ 1) is a measure of momentum resistance due to the outflow, and t is the time coordinate

If we ignore the outflow function Ψ and the frictional term f equations 1 and 2 are similar to the one dimensional hyperbolic equations of gas dynamics. Hence shock may develop in the solutions . (Sachedev et al, 1997)

The functions Ψ and f are chosen to be known functions of A, u, P, x, and t so equations 1 and 2 are two differential equations in three unknowns: A, u, P. Therefore a third equation to complement them is needed . This is the state equation. For a purely elastic model, the state equation simply relates A and P which holds under static condition.

$$P = F(x, A) \qquad (3)$$

However, for a viscoelastic material as our model proposes, the strain depends upon the magnitude of the stress and the rate at which it is applied. Hence the state equation should describe the response of the wall to the internal pressure. It should relate the lumen area to the instantaneous pressure and the time rate of pressure. Therefore we proposed the state equation as:

$$P = P(x, A, \partial A/\partial t) \quad (4)$$

where the dependency on x, A, $\partial A/\partial t$ measures the non uniformity (tapering), elasticity and viscoelasticity of the tube respectively

We now propose two models of determining the explicit form of equation 3.4 and to measure the effect of the viscoelasticity of the tube on wave propagation and pressure pulse in the arterial system.

In choosing a form for the equation of state, we adopt Fung (1968) method. He represented the history of the stress response called the relaxation constant $k(\lambda,t)$ by

$$k(\lambda,t) = G(t)\,T^{(e)}(\lambda), \quad G(0) = 1 \quad (5)$$

where λ is the strain, $G(t)$ a normalised function of time is the reduced relaxation function and $T^{(e)}(\lambda)$ is called the elastic response. He then assumed that the stress response to an infinitesimal change in stretch $\delta\lambda(\tau)$ superposed on a specimen in a state of stretch λ at an instant time of τ is for $t > \tau$:

$$G(t-\tau)\frac{\partial T^{(e)}}{\partial \lambda}[\lambda(\tau)]\delta\lambda(\tau) \quad (6)$$

Thus the tensile stress at a time t

$$T(t) = \int_{-\infty}^{t} G(t-\tau)\dot{T}^{(e)}(\tau)d\tau \quad (7)$$

This implies that the stress response is described by a linear law relating stress T with the elastic response $T^{(e)}$. The function $T^{(e)}(\lambda)$ plays the role assumed by the strain in the conventional theory of elasticity. Finally he deduced that

$$T(t) = T^{(e)}[\lambda(t)] + \int_0^t T^{(e)}[\lambda(t-\tau)]\frac{\partial G(\tau)}{\partial \tau}d\tau \quad (8)$$

i.e. the tensile stress at any time t is equal to the instantaneous stress response $T^{(e)}[\lambda(t)]$ decreased by an amount depending on the past history because $\frac{\partial G(\tau)}{\partial \tau}$ is generally of negative value.

Based on this linear theory, we assume the state equation in the form

$$P = F(x,A) + G\left(A, \frac{\partial A}{\partial t}\right) \qquad (9)$$

where the function F(x, A) corresponds to the static loading and the function G(A,∂A/∂t) accounts for the viscoelasticity properties of the wall. G is a monotonically increasing function of its second argument with G(A,0) = 0. (Kivity & Collins 1974 a,b).

In this work, we are particularly interested in the influence of viscoelastic term G on the pressure and velocity pulses obtained from calculation based on elastic model. Comparison with measurements may then afford an impression of the size of the viscoelastic component in the constitutive equation.

To gain a good insight, we consider a flow in a uniform tube and neglect fluid viscosity and outflow from the tube. Equations 1 and 2 become.

$$\frac{\partial A}{\partial t} + \frac{\partial (Au)}{\partial x} = 0 \qquad (10)$$

$$\frac{\partial u}{\partial u} + u\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0 \qquad (11)$$

which combine to give

$$\frac{\partial (Au)}{\partial t} + \frac{\partial (Au^2)}{\partial x} + \frac{A}{\rho}\frac{\partial P}{\partial x} = 0 \qquad (12)$$

Let Q = Au (flow rate) then the equations 10 and 12 become.

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (13)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial Q^2/A}{\partial x} + \frac{A}{\rho}\frac{\partial P}{\partial x} = 0 \qquad (14)$$

If G possesses a double Taylor series expansion about the point (A,0), the first two terms vanish owing to the condition G(A,0) = 0. Then the lowest order non-vanishing term turns out to be proportional to ∂A/∂t Thus to a first level of approximation, G can be of the form

$$g(A, \partial A/\partial t) = \frac{k}{A}\frac{\partial A}{\partial t} \tag{15}$$

Here k is a measure of the influence of viscoelasticity. Thus equation 9 becomes

$$P = F(A) + \frac{k}{A}\frac{\partial A}{\partial t} \tag{16}$$

The effect of the elasticity of the tube measured by F(A) has been established over and over (Rudinger 1968, Anliker et al 1971, Smit 1981 Hoogstraten & Smit 1981). We now concentrate on the effect of the viscoelastic factor by ignoring F(A).

Using equation 3.16 in equation 3.11 we have.

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial t}\left(\frac{Q^2}{A}\right) + \frac{k}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial t}\right) = 0 \tag{17}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(Q^2/A) - \frac{k}{\rho}\frac{\partial^2 Q}{\partial x^2} = 0 \tag{18}$$

Using equation 3.10 then we have

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} - \frac{k}{\rho}\frac{\partial^2 u}{\partial x^2} = 0 \tag{19}$$

If we ignore the non-linear term, equation 19 is the parabolic diffusion equation with diffusivity constant $\nu = k/\rho$ which is a measure of the viscoelasticity. Thus we see that the introduction of viscoelasticity into the tube introduces a retarding effect in the constituent equations. Also equation 19 is a form of Burger's equation except for the coefficient 2 in the non-linear term. It shows, like Burger's equation, an interplay between the non-linear steeping and the diffusion of a wave. Thus shocklike transition may develop in the solution in the form of very steep but continuous wave front.

ANALYTIC METHOD OF SOLUTION.

We now attempt solution to equation 19 in the form

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} - \upsilon\frac{\partial^2 u}{\partial x^2} = 0 \tag{20}$$

Let $u = \partial\Phi/\partial x$, $\Phi = \Phi\ (x, t)$ (21)

Then equation 20 transform into

$$\frac{\partial^2\Phi}{\partial x\partial t}+2\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2}-\upsilon\frac{\partial^3\Phi}{\partial x^3}=0 \qquad (22)$$

Integrating with respect to x and ignoring the constant of integration, we have

$$\frac{\partial\Phi}{\partial t}+(\frac{\partial\Phi}{\partial x})^2-\upsilon\frac{\partial^2\Phi}{\partial x^2}=0 \qquad (23)$$

Equation 23 is invariant under the transformation

$x\to ax, \quad t\to a^2t \quad (a = \text{constant}) \qquad (24)$

as in the diffusion equation. This suggests that there exists a solution of the form.

$\Phi(x, t) = \Phi(\Omega(x, t)) \qquad (25)$

where Ω is a solution of the diffusion equation

$$\frac{\partial\Omega}{\partial t}-\upsilon\frac{\partial^2\Omega}{\partial x^2}=0 \qquad (26)$$

However, equation 20 is not satisfied by the Cole-Hopf transformation of Burger equation (Hopf, 1950).

$$\Phi=-2\upsilon\log_e\Omega \qquad (27)$$

We however attempt a similar solution based on the non-invariance of equation 23 by substituting 25 into 23 we have

$$\frac{\partial\Omega}{\partial t}\frac{\partial\Phi}{\partial\Omega}+(\frac{\partial\Omega}{\partial x}\frac{\partial\Phi}{\partial\Omega})^2-\upsilon(\frac{\partial\Phi}{\partial\Omega}\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Phi}{\partial\Omega^2}(\frac{\partial\Omega}{\partial x})^2)=0 \qquad (28)$$

Using 26 we have

$$(\frac{\partial\Phi}{\partial\Omega})^2(\frac{\partial\Omega}{\partial x})^2-\upsilon\frac{\partial^2\Phi}{\partial\Omega^2}(\frac{\partial\Omega}{\partial x})^2=0 \qquad (29)$$

$$\left(\frac{\partial\Phi}{\partial\Omega}\right)^2-\upsilon\frac{\partial^2\Phi}{\partial\Omega^2}=0 \qquad (30)$$

The solution of this equation is given by

$\Phi(\Omega) = -\upsilon\log_e(\Omega - a) + b \qquad (31)$

where a, b are constants. By choosing a = 0 we have

$$u(x,t)=-\frac{\upsilon}{\Omega}\frac{\partial\Omega}{\partial x} \qquad (32)$$

The non-viscoelastic solution of equation 26, which satisfies equation 20 in the limit $\upsilon\to 0$, is given as:

$$\Omega = \frac{1}{2\upsilon t}\exp(-\frac{x^2}{4\upsilon t}) \quad (33)$$

$$u = \frac{x}{2t} \quad (34)$$

Whatever mathematical model we are proposing must be realistic and close to its modeled end. Hence we choose the cylindrical coordinates (r,θ,z) with axisymmetry for our solution. Thus equation 3.20 in these coordinates becomes

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial r} = \upsilon[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - \frac{u}{r^2}] \quad (35)$$

$$= \upsilon\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(ru) \quad (36)$$

Nervey et al (1996) had proved using the vector Burger's equation that the solution to the Burger's equation can be generalised to n-dimension. We thus employ this to derive solution to equation 35 in the form

$$u = -\frac{\upsilon}{\Omega}\frac{\partial\Omega}{\partial r} \quad (37)$$

where the diffusion equation is

$$\frac{\partial\Omega}{\partial t} - \frac{\upsilon}{r}\frac{\partial}{\partial r}(r\frac{\partial\Omega}{\partial r}) \quad (38)$$

The non-viscoelastic equation is given as

$$\Omega = \frac{1}{2\upsilon t}\exp(\frac{-r^2}{4\upsilon t}) \quad (39)$$

$$u = \frac{r}{2t} \quad (40)$$

However it is important that we derive the solution of the diffusion equation that are based on physical boundary conditions on the velocity field. To this end we formally integrate equation 35 to have.

$$\Omega(r,t) = k(t)\exp\left[-\frac{1}{\upsilon}\int_0^r u(w,t)dw\right]$$

$$k(t) = \Omega(0,t) \quad (41)$$

The velocity is to be initially specified over a given range so that the initial value of Ω is given by

$$\Omega(r,0)=\Omega_0(r)=k_0\exp\left[-\frac{1}{\upsilon}\int u_0(w)dw\right] \quad (42)$$

for $r_0 \le r \le R$.

The general solution of the diffusion equation 38 with azimuthal symmetry according to Nerver et al (1996) can now be written as:

$$\Omega(r,t)=\int_0^\infty\int_0^\infty \Omega_0(r')J_0(kr')J_0(kr)\exp(-\upsilon k^2 t)kr'dkdr' \quad (43)$$

The integral over k can be done since it is a special case of Weber's second exponential integral (Watson, 1995; Hanson and Puja 1997). According to these authors

$$\int_0^\infty \exp\left(-p^2t^2\right)J_\vartheta(\alpha t)J_\vartheta(\beta t)tdt=\frac{1}{2p^2}\exp\left(-\frac{a^2+b^2}{4p^2}\right)I_\vartheta\left(\frac{\alpha\beta}{2p^2}\right) \quad (44)$$

$$\int_0^\infty \left[\exp(-\upsilon tk^2)J_0(kr)J_0(kr')k\right]dk=\frac{1}{2\upsilon t}\exp-\left(\frac{r^2+r'^2}{4\upsilon t}\right)I_0\left(\frac{rr'}{2\upsilon t}\right) \quad (45)$$

where the modified Bessel function of order zero may be written as

$$I_0\ (ar') = J_0(iar') \quad (46)$$

which is a real function. Equation 43 can now be written as

$$\Omega(r,t)=\frac{1}{2\upsilon t}\exp\left(\frac{-r^2}{4\upsilon t}\right)\int_0^a[\Omega_0(r')\exp\left(\frac{-r'}{4\upsilon t}\right)J_0\left(\frac{irr'}{2\upsilon t}\right)r']dr' \quad (47)$$

We now specify an incompressible fluid flowing from the origin of coordinates and spreading out in a circular symmetric pattern. We also assume that the speed must decrease as r to conserve mass in an initially steady flow i.e.

$$u(r,0)=\frac{u_0r_0}{r}, \qquad r_0\le r\le R \quad (48)$$

so that equation 42 becomes.

$$\Omega_0=k_0\left(\frac{r_0}{r}\right)^a \quad (49)$$

$$a = \frac{u_0 r_0}{2\upsilon} \qquad (50)$$

Clearly u_0 may be large at the small value r_0 in such a way that the product $u_0 r_0$ is well defined. Equation 47 can now be integrated using Hankel's generalisation of Weber's second order exponential integral

$$\Omega(r,t) = \frac{1}{2\upsilon t}\exp - \frac{r^2}{4\Omega t} k_o r_o^a \int \exp\left(-\frac{r'^2}{4\upsilon t}\right) J_O\left(\frac{irr'}{2\upsilon t}\right) r'^{1-a} dr' \qquad (51)$$

$$\Omega(r,t) = k_0 r_0^a \Gamma\left(1 - \tfrac{a}{2}\right)(4\upsilon t)^{-a/2} \sum_0^\infty \frac{\left(\frac{a}{2}\right)_n}{(n!)^2}\left(\frac{-r^2}{4\upsilon t}\right)^n \qquad (52)$$

where $(a/2)_n$ is the Pochhammer notation:

$$\left(\frac{a}{2}\right)_n = \frac{a}{2}\left(\frac{a}{2}+1\right)\left(\frac{a}{2}+2\right)\ldots\ldots\left(\frac{a}{2}+n-1\right) \qquad (53)$$

$$\left(\frac{a}{2}\right)_0 = 1$$

The full non linear solution can now be derived from equation 36 and expressed in terms of the confluent hypergeometric function as

$$u(r,t) = \frac{a\,r}{4\,t}\frac{M\left(\frac{a}{2}+1, 2, -\frac{r^2}{4\upsilon t}\right)}{M\left(\frac{a}{2}, 1, -\frac{r^2}{4\upsilon t}\right)} \qquad (54)$$

$$= \frac{r}{2t}\frac{\sum_0^\infty \left(\frac{a}{2}\right)_n \frac{n}{(n!)^2}\left(\frac{-r^2}{4\upsilon t}\right)^{n-1}}{\sum_0^\infty \left(\frac{a}{2}\right)_n \frac{1}{(n!)^2}\left(\frac{-r^2}{4\upsilon t}\right)^n} \qquad (55)$$

If we dimensionalise equation by the transformation.

$$r \to r/L, \qquad t \to \upsilon t/l^2, \qquad u \to u/U. \qquad (56)$$

Equation 35 becomes

$$\frac{\partial u}{\partial t}+2\left(\frac{Lu}{\upsilon}\right)u\frac{\partial u}{\partial r}=\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial l}\right)-\frac{u}{r^2}\right] \tag{57}$$

If we choose $U = u_0$, $L = r_0$ we then have

$$\left(\frac{Lu}{\upsilon}\right)=1 \tag{58}$$

from 50 a = 0.5

DISCUSSIONS

The introduction of viscoelasticity through equation 15 destroys the hyperbolic nature of the equation. Hence discontinuities in the solutions are no more admissible, however shock like transition may develop in the form of steep but continuous wave fronts.

We see that the solution can be written as the product of the inviscid solution times an infinite sum of powers of the similarity variable $(r^2/4vt)$. The effect of the non-linearity on higher powers of the similarity variable can be seen through the occurrence of the factor n in the numerator. The transient non-linearity steepens but diffusion dominates. (The transient is not by itself a solution of our equation since we do not have the superposition theorem). This agrees well with Pedley (1981) that the effect of the non-linearity in such an inviscid uniform tube model is to cause the front of any wave to steepen because wave speed increases with u. However, rather than this steepen creating discontinuity, diffusion introduced by viscoelasticiy predominates. Smit (1982) has shown that under physiologically normal circumstances pressure recording taken at various site along the aorta shows sharp peaks but real discontinuities are not recorded.

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