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VARIABLE VISCOSITY REACTING FLOWS WITH CONSERVED ENERGY

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ABSTRACT:

A premixed endothermic reacting viscous flow with conserved energy is presented. Despite the integral constraint $\int_{-1}^{1} u(x,t)dx = 0$ we establish existence of solution via the method of upper and lower solutions

1. INTRODUCTION:

In a recent paper, Budd el al. [1] studied the problem

$$u_t = u_{xx} + u^2 - k^2 (t) \tag{1}$$

$$u_x (0, t) = u_x (1, t) = 0 \tag{2}$$

$$u (x, 0) = u_0 (x) \tag{3}$$

together with the integral constraint

$$\int_0^1 u(x,t)dx = 0 \tag{4}$$

The problem thus has two boundary conditions and integral constraint. To compensate for this, the function k(t) was regarded as unknown and was determined as part of the solution.

In this paper k(t) is due to the endothermic reaction and hence known. In particular we consider a channel reacting flow; that is, a problem defined on the interval [-h, h]. Here we consider the existence of solution and in a subsequent paper we investigate the properties of the solution.

2. MATHEMATICAL FORMULATION

Momentum equation

$$0 = (\mu V_y)_y \tag{5}$$

where

$$\mu = \mu_0 \exp -\alpha (T - T_0) \tag{6}$$

Consumption of the reactants

$$Y_t = DY_{yy} - AYF(T) \tag{7}$$

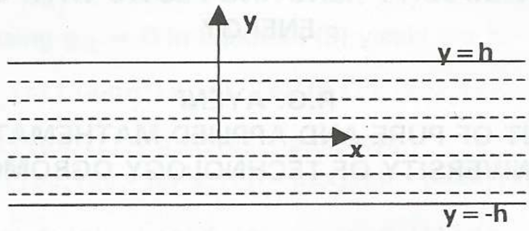

Fig 1. A reacting flow in a channel.

Energy equation

$$\rho\delta T_t = DY_{yy} + \mu(V_y)^2 - QAYf(T) \quad (8)$$

where P, v, T and Y denote pressure, velocity temperature and premixed reactants respectively. μ is viscosity. D is diffusion coefficient, A is pre-exponential factor, α is a positive constant and δ is the specific heat. The function f depends on the reaction . For example, the gas phase reaction

$$\tfrac{1}{2}\ N_2 + \tfrac{3}{2} H_2 = NH_3 \quad (9)$$

that is

$$v_1 N + v_2 H_2 + v_3 NH_3 = -\tfrac{1}{2} N_2 - \tfrac{3}{2}\ H_2 + NH_3 = 0 \quad (10)$$

has

$$f(T) = \exp[-\sum_{i=1}^{3} \frac{v_i}{RT} \int_0^{p_0} (v_i - \frac{RT}{p}) dp] \quad (11)$$

Here we take

$$f(T) = \int_1^1 \exp - \frac{E}{RT}\ dy \quad (12)$$

On using (5) and (6) in (8) we obtain

$$T_t = T_{yy} + B\ \exp(T \quad T)\ \mu \quad QAY \int_{-1}^{1} \exp - \frac{1}{RT}\ dy \quad (13)$$

where B is a constant, Q is the heat release per unit mass, E is the activation energy and R is the universal gas constant.
Equation (13) is non-dimensionalized by the transformations

$$\tau = t/h, \quad z = \frac{y}{h}, \quad u = (T - T_o)\, E/RT_o^2 \qquad (14)$$

to obtain

$$u_\tau = a u_{zz} + b\exp(Bu) - c\int_{-1}^{1} \exp(u/1+u)\, dz \qquad (15)$$

where,

$$a = \lambda/\rho\,\delta Uh, \quad b = B^2h/\mu_o\,\rho\delta U$$

$$c = QAhY\exp(E/2RT_o)/\rho\delta\, U, \quad E = RT_o/E$$

$$\beta = \alpha\, RT_o^2/E \qquad (16)$$

where $v = U$ at $y = h$ (17)

It suffices in this paper to consider a = b = c = 1, other values of these parameters will be discussed in another paper. We shall consider the following cases

(a) $\varepsilon \to 0, \ \beta = 1$. In this case (15) becomes

$$u_\tau = u_{zz} + \exp u - \int_{-1}^{1} \exp u\, dz \qquad (18)$$

(b) $\varepsilon^2 \to 0, \ \beta^2 = 1$. In this case (15) becomes

$$u_\tau = u_{zz} + \exp u - \int_{-1}^{1} \exp u\,(1 - \varepsilon u)\, dz \qquad (19)$$

(c) $\varepsilon^2 \to 0, \ \beta^2 \to 0$. In this case (15) becomes

$$u_\tau = u_{zz} + 1 + \beta u - \int_{-1}^{1} \exp u\,(1 - \varepsilon u)\, dz \qquad (20)$$

(d) $\varepsilon \to 0, \ \beta^2 \to 0$. In this case (15) becomes

$$u_\tau = u_{zz} + 1 + \beta u - \int_{-1}^{1} \exp u\, dy \qquad (21)$$

(e) $\varepsilon = \beta = 1$. In this case (15) becomes

$$u_\tau = u_{zz} + \exp u - \int_{-1}^{1} \exp(u/(1+u))\, dy \qquad (22)$$

We shall investigate the conditions for existence of solutions of (a) – (e) under the initial and boundary conditions

$$U(z, 0) = u_0(z) \tag{23}$$
$$U_z(-1, t) = u_z(1, t) = 0 \tag{24}$$
$$\int_{-1}^{1} u(z, t)\, dz = 0 \tag{25}$$

3. EXISTENCE OF SOLUTION

Consider the problem

$$-u_t + u_{zz} + f(z, u) = 0 \tag{26}$$

satisfying (23) and (24)

DEFINITION: A smooth function $u_2(x, t)$ is said to be an upper solution of (26) if

$$-u_{2t} + u_{2zz} + f(z, u_2) \leq 0 \tag{27}$$
$$u_2(z, 0) \geq u_0(z) \tag{28}$$
$$u_{2z}(-1, t) \geq 0, \; u_{2z}(1, t) \geq 0 \tag{29}$$

Similarly, $u_1(x,t)$ is called a lower solution of (24) if $u_1(x, t)$ satisfies the reversed inequalities (27) – (29).

REMARK 2: Upper and lower solutions for (26) which satisfy (23) – (25) are impossible to establish. For if $u_1(z, \tau) < u_2(z, \tau)$ the integral constraint

$$0 = \int_{-1}^{1} u_1(z, \tau)\, dz < \int_{-1}^{1} u_2(z, \tau)\, dz = 0$$

leads to a contradiction.

In this paper we seek a solution of the form

$$u(z, \tau) = h(z)\, v(\tau) \tag{30}$$

such that

$$\int_{-1}^{1} h(z)\, dz = 0 \tag{31}$$
$$h(z) = u_0(z) \tag{32}$$
$$h_z(-1) = h_z(+1) = 0 \tag{33}$$

for example we consider

$$u_0(z) = \lambda \cos \pi z \tag{34}$$

for some $\lambda > 0$

clearly for

$$u(z, \tau) = h(z)\, v(\tau) = \lambda \cos \pi z v(\tau) \tag{35}$$

(31) and (33) are automatically satisfied.
Let us consider a test case the problem

$$U_\tau = u_{zz} + u^2 \int_{-1}^{1} u^2 \, dz, \qquad z \, \varepsilon(-1, 1) \qquad (36)$$

Such that u satisfies (23) – (24)
We seek

$$U(z, \tau) = h(z)\, v(\tau) = \cos \pi z v(\tau) \qquad (37)$$

$$U_0(z) = h(z), \; v(0) = 1 \qquad (38)$$

Substituting (37) into (36) we obtain

$$v_\tau = -\lambda\pi^2 \cdot v - \lambda^2 v^2 \tan \pi z \sin \pi z \qquad (39)$$

Hence (39) and (38) imply

$$v = \frac{1}{(1+\frac{\lambda}{\pi^2}\tan \pi z \sin \pi z)\, e^{\lambda\pi^2\tau} - \frac{\lambda}{\pi^2}\tan \pi z \sin \pi z} \qquad (40)$$

and u(z, t) = cos z v (τ)
Clearly u (z, τ) blows up for

$$e^{\lambda\pi^2\tau} = \frac{\frac{\lambda}{\pi^2}\tan \pi z \sin \pi z}{1+\frac{\lambda}{\pi 2}\tan \pi z \;\; \sin \pi z} \qquad (41)$$

CASE (A)

$$U_\tau = u_{zz} + \exp u - \int_{-1}^{1} \exp u \, dz$$

Now

$$U = h(z)\, v(\tau) = \lambda \cos \pi z v(\tau) \qquad (42)$$

$$U_0(z) = h(z), \quad v(0) = 1 \qquad (43)$$

implies

$$\lambda \cos \pi z v_\tau = -\lambda\pi^2 v + \exp(\lambda \cos \pi z v)$$

$$- \int_{-l}^{l} \exp(\lambda \cos \pi zv)\, dz \qquad (44)$$

i.e. $-\lambda\pi^2 v + \exp(\lambda \cos \pi zv) - \int_{-1}^{1} \exp \lambda \cos \pi zv - \lambda \cos \pi z v_\tau = 0$ (45)

Clearly there exists a solution of (45) which satisfies $v(0) = 1$ for some values of λ. For example for $\lambda = 1$

$v_1 \equiv 0$ is a lower solution and $v_2 \equiv 1$ is an upper solution.

4 CONCLUSION:

The full behaviour of a viscous reacting flow will be investigated in another paper. The cases (b) – (e) will be examined. In this paper we provide an analytical solution for a problem considered in [1]. We also show that there exists a solution of a reacting viscous flow when certain criteria are met.

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