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A CONTROL OPERATOR FOR A CLASS OF REGULATOR PROBLEMS[1,2]

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ABSTRACT

The continuous optimal control problem characterized by linear system integral costs had been solved successfully via the Extended Conjugate Gradient Method (ECGM). A Control Operator for this class of problems, which improves the overall performance of the ECGM algorithm is constructed herein. Three numerical examples are reported in support of the conclusions of this paper.

Key Words: Control Operator, Regulator Problems, Conjugate Gradient, Functional.

1. INTRODUCTION

The philosophy of the Conjugate Gradient Method (CGM) algorithm for solving problems of the form:

$$\text{Min } F(x) = \text{Min } \{f_0 + \langle a, x \rangle_H + \tfrac{1}{2} \langle x, Ax \rangle_H \} \tag{1}$$

where A is a linear symmetric, positive definite operator on the Hilbert Space H, a, $x \in H$ and $\langle .,. \rangle_H$ is the usual inner product on the Hilbert space H; had been extended to handle the minimization of penalized cost functionals for control problems characterized by linear-system integral cost, of the type given by

Problem (1):

$$\text{Min } J(X,U) = \text{Min} \int_0^{\sigma} [X^T(t)QX(t) + U^T(t)RU(t)]dt \tag{2}$$

Subject to the dynamic constraint

$$X'(t) = CX(t) + DU(t), \; 0 \le t \le \sigma, \tag{3}$$
$$X(0) = X_0 \quad , (\sigma \text{ specified}).$$

In Equations (2) and (3), X(t) is the state of the system at time t,U(t) is the control applied to the system at time t, $0 \le t \le \sigma$. σ is a specified final time. C and B are square, constant matrices of order n, while Q and R are symmetric,

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The continuous optimal control problem of interest here is that of finding an optimal control function U*(.), defined on the closed interval [0,σ], together with a corresponding trajectory X*(.), determined by Equation (3), which minimizes Equation (2). The authors of Ref.1 , while adopting a penalty optimization technique for this class of problems, proposed the Extended Conjugate Gradient Method, (ECGM), by associating Problem (1) with a control operator A, such that:

$$\operatorname{Min} J(X,U,\mu) = \operatorname{Min}\langle Z, AZ\rangle_K = \operatorname*{Min}_{(X,U)} \{ \int_0^\sigma [X^T(t)QX(t) + U^T(t)RU(t)]\,dt + \mu \int_0^\sigma \| \dot{X}(t) - CX(t) - DU(t)\|^2 dt \} \quad (4)$$

In Equation (4), K is the cartesian product space of $H_1[0,\sigma]$ and $L_2^n[0,\sigma]$. $H_1[0,\sigma]$ is the Sobolev space of absolutely continuous functions X(.), square integrable with first derivatives also square integrable over the specified interval [0,σ]. $L^n_2[0,\sigma]$ is the Hilbert space, consisting of equivalence classes of square integrable functions from [0,σ] into IR^n. The Space K is endowed with the norm and inner product given respectively as follows

$$\| Z\|_k^2 = \| X\|^2_{L_2} + \| \dot{X}\|^2_{L_2} + \| U\|^2_{L_2}, \quad (5)$$

$$Z = (X,U)^T, \; X \in L_2[0,\sigma], \; U \in L_2[0,\sigma];$$

$$\langle .,.\rangle_K = \langle .,.\rangle_{H_1} + \langle .,.\rangle_{L_2}; \quad (6)$$

with the norm defined by

$$\| U\| = \{ \int_0^\sigma \| U(t)\|_E^2 dt \}^{\frac{1}{2}}$$

and inner product

$$\langle U_1, U_2\rangle = \int_0^\sigma \langle U_1(t), U_2(t)\rangle_E dt$$

where $\|.\|_E$ and $\langle .,.\rangle_E$ denote the norm and scalar product in the Euclidean n-dimensional space.

Our intent in this paper is to construct a new control operator, G, in a way akin to the operator A, of Ref. 2, which when substituted for A, in the ECGM algorithm, improves the performance of the scheme.

In what immediately follows, we briefly outline the Extended Conjugate Gradient Method (ECGM) algorithm. Thereafter, we proceed to construct the desired operator G.

2. THE EXTENDED CONJUGATE GRADIENT METHOD (ECGM) ALGORITHM.

The explicit determination of the control operator A satisfying Equation (4), permits the extension of the computational scope of the Conjugate Gradient Method (CGM) algorithm, given in Ref. 3, to a broader class of problems such as Problem (1) while retaining its simplicity and elegance.

In order to determine the control operator A that satisfies Equation (4), the authors of Refs. 2 and 4, considered a one-dimensional control problem, whose associated control operator is a special case of that associated with Problem (1). Furthermore, Ref. 1 generalized the construction of operator A and proposed the Extended Conjugate Gradient Method (ECGM) algorithm, which is defined in the following manner.

The ECGM Algorithm:(Ref. 1).

Guess $Z(0) = (X(0), U(0))^T \in H.$ *and compute*

$$\begin{aligned} p_{X,0} &= -g_{X,0}, & p_{U,0} &= -g_{U,0} \\ X_{i+1} &= X_i + \alpha_i p_{X,i}; & U_{i+1} &= U_i + \alpha_i p_{u,i} \\ g_{X,i+1} &= g_{X,i} + \alpha_i A P_{X,i}; & g_{U,i+1} &= g_{U,i} + \alpha_i A P_{U,i} \\ P_{X,i+1} &= -g_{X,i+1} + \beta_i P_{X,i}; & P_{U,i+1} &= -g_{U,i+1} + \beta_i P_{U,i} \end{aligned} \tag{7}$$

where

$$\alpha_i = \frac{\langle g_i, g_i \rangle}{\langle P_i, AP_i \rangle}, \quad \beta_i = \frac{\langle g_{i+1}, g_{i+1} \rangle}{\langle g_i, g_i \rangle} \text{ and } \quad g_i = \begin{pmatrix} g_{u,i} \\ g_{x,i} \end{pmatrix}, \quad P_i = \begin{pmatrix} p_{x,i} \\ p_{u,i} \end{pmatrix},$$

$$AP_i = -\mu[J_{x,0} - cP_{x,0}]\sinh t - \mu\int_0^t (J_{x,i} - cP_{x,i})\cosh(t-s)\,ds$$

$$-\mu\int_0^t [(a+\mu c^2)P_{x,i} - \mu c J_{x,i}]\sinh(t-s)\,ds + [(a+\mu c^2)P_{x,0} - \mu c J_{x,0}]\cosh t$$

$$+\frac{\sinh t}{\sinh\sigma}\{(a+\mu c^2)x(\sigma) - \mu c\dot{x}(\sigma) - \mu\sinh\sigma[J_{x,0} - cP_{x,0}] - \mu\int_0^\sigma [J_{x,0} - cP_{x,0}]$$

$$\cosh(\sigma-s)\,ds + \int_0^\sigma [(a+\mu c^2)P_{x,i} - \mu c j_{x,i}]\sinh(\sigma-s)\,ds + [(a+\mu c^2)P_{x,0} - \mu c J_{x,0}]$$

$$\cosh\sigma\} + \mu du(0)\sinh t - \mu d\int_0^t u(s)\cosh(t-s)\,ds + \mu c d\int_0^t u(s)\sinh(t-s)\,ds$$

$$+\mu c d u(0)\cosh t + \frac{\sinh t}{\sinh\sigma}\{\mu c d u(\sigma) - \mu d u(0)\sinh\sigma + \mu\int_0^\sigma d\dot{u}(s)\cosh(\sigma-s)\,ds$$

$$+\mu c d\int_0^\sigma u(s)\sinh(\sigma-s)\,ds - \mu c d u(0)\cosh\sigma\}, \; cdP_{x,i} - dJ_{x,i} + (b+d^2)u(t). \tag{8}$$

In Equations (7) and (8), the following "abused" notations have been adopted:

$J_{x,i} = J_x(x_i\ \ u_i,\mu)$, $\quad J_{u,i} = J_U(x_i,\ u_i,\mu)$, $\ P_{x,i} = P_x(x_i,\ u_i,\mu)$,

$P_{u,i} = P_u(x_i,\ u_i,\mu)$, $J_i = J(x_i,\ u_i,\mu)$,

$$P_x(x_i(t), u_i(t), \mu) = \int_0^t J_x(x_i(s), u_i(s), \mu)\,ds$$

and $\quad P_u(x_i(t), u_i(t), \mu) = \int_0^t J_U(x_i(s), u_i(s), \mu)\,ds.$

Furthermore, g_i, α_1, and P_1 denote respectively the gradient of $J(x,u)$, the step length of the descent sequence and the descent direction at the ith step.

Evidently, the explicit determination of the Operator A, satisfying Equation (4), which gave rise to Equation (8) eliminated the computational limitations of the Conjugate Gradient Method of Ref. 3 and gave way to the proposition of the Extended Conjugate Gradient Method (ECGM) algorithm; which is computationally more exact and less cumbersome than the conventional function space algorithm reported in Ref. 5. These advantages were confirmed by Ref. 6. Other applications of the Operator A have been considered by the authors of Ref. 7.

The remainder of the paper consists of three sections. Section 3 is devoted to the construction of the control operator G which improves the efficiency of the Extended Conjugate Gradient Method. Section 4 presents numerical examples with the aim of illustrating the effectiveness of the constructed operator. Finally, section 5 contains some concluding remarks and discussions.

3. The Construction of Operator G.

Here, we provide the mathematical tools necessary for the construction of operator G. We start by considering the one-dimensional control problem:

Problem (2):

$$Min\ F(x,u) = Min\int_0^\sigma [ax^2(t) + bu^2(t)]\,dt$$

subject to the dynamic constraint

$$\dot{x}(t) = cx(t) + du(t), \quad 0 \le t \le \sigma \quad x(0) = x_0,$$

where a and b are positive constants, while c and d are specified constants not necessarily positive.

Associated with the constrained Problem (2) is the unconstrained problem given by

Problem (3)

$$MinF(x,u,\mu) = min\int_0^\sigma [ax^2(t) + bu^2(t) + \mu(\dot{x}(t) - cx(t) - du(t))^2]\,dt.$$

Let $w_1 = (x_1,u_1)$ and $w_2 = (x_2,u_2)$, then it is convenient to associate with $F(x,u,\mu)$ the following bilinear form

$$R_\mu(w_1,w_2) = \int_0^\sigma [ax_1(t)x_2(t) + bu_1(t)u_2 + \mu\dot{x}_1(t)\dot{x}_2(t) + \mu c^2x_1(t)x_2(t) + \mu d^2u_1(t)u_2 - \mu cx_1(t)x_2(t). + \mu cdx_1(t)u_2(t) - \mu c\dot{x}_2(t)x_1(t) - \mu d\dot{x}_{.}(t)u_1(t) + \mu cdx_2(t)u_1(t) - \mu dx_1(t)u_2(t)]dt. \tag{9}$$

It is proved in Ref. 2 that $R_\mu(w_1,w_2)$ given by Equation (9) is a bounded, bilinear, self-adjoint Hermitian form. Furthermore, the authors of Ref. 8 showed that every self-adjoint bounded operator generates a bounded bilinear Hermitian form and conversely, ifa bounded Hermitian form is given,it defines a bounded self-adjoint operator.

Following these results, we associate with the form $R_\mu(w_1,w_2)$ the control operator G, which satisfies the requirements;

$$\langle g_1(t), Gg_2(t)\rangle_k = \int_0^\sigma [ax_1(t)x_2(t) + bu_1(t)u_2(t) + \mu\dot{x}_1(t)\dot{x}_2(t) + \mu c^2x_1(t)x_2 + \mu d^2u_1(t)u_2(t) - \mu cx_1(t)x_2(t) + \mu cdx_1(t)u_2(t) - \mu c\dot{x}_2(t)x_1(t) - \mu d\dot{x}_2(t)u_1(t) + \mu cdx_{.}(t)u_1(t) - \mu dx_1(t)u_2(t)]dt \tag{10}$$

In Equation (10), we have adopted the following notations;

$$g(t) = (g_x(t),\ g_u(t))^T$$

$$g_i(t) = (\dot{x}_i(t), \dot{u}_i(t))^T = (g_{x_i}(t), g_{u_i}(t))^T = (g_{x_i}, g_{u_i})^T;\ i=1,2,\quad 0 \le t \le \sigma.$$

Thus, the operator G is defined as follows:

$$Gg_2 = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} g_{x_2} \\ g_{u_2} \end{bmatrix} = \begin{bmatrix} G_{11}g_{x_2}(t) + G_{12}g_{u_2}(t) \\ G_{21}g_{x_2}(t) + G_{22}g_{u_2}(t) \end{bmatrix} \tag{11}$$

where G_{11}, G_{12}, G_{21} and G_{22} are operators to be determined.

The Riesz representation theorem guarantees the existence of operator G satisfying (10), since the right hand side of (10) contains a bilinear, self-adjoint and continuous form, Ref. 9.

Now suppose $g_{u2} = 0$, then Equation (11) reduces to

$$Gg_2 = \begin{bmatrix} G_{11}g_{x_2} \\ G_{21}g_{x_2} \end{bmatrix} = \begin{bmatrix} \hat{g}_{x_2} \\ \hat{g}_{u_2} \end{bmatrix} \quad (12)$$

Furthermore, using $g_{u2} = 0$ in Equation (9) yields the functional given by

$$\begin{aligned} <g_1, zg_2> &= \int_0^\sigma \{x_1(t)[(a+\mu c^2)x_2(t) + \mu c d u_2(t) - \mu c \dot{x}_2(t)] \\ &+ \dot{x}_1(t)[\mu \dot{x}_2(t) - \mu c x_2(t) - \mu d u_2(t)] \\ &+ u_1(t)[b u_2(t) + \mu d^2 u_2(t) - \mu d \dot{x}_2(t) + \mu c d x_2(t)] \\ &= \int_0^\sigma [x_1(t)\hat{g}_{x_2}(t) + \dot{x}_1(t)\hat{g}_{x_2}(t) + u_1(t)\hat{g}_{u_2}(t)]dt. \end{aligned} \quad (13)$$

To obtain the operators G_{11} and G_{21} given by Equation (12) we must uniquely determine $g_{x2}(t)$ and $g_{u2}(t)$; to achieve that we let

$$\alpha(t) = (a+\mu c^2)x_2(t) + \mu c d u_2(t) - \mu c \dot{x}_2(t), \quad (14a)$$

$$\beta(t) = \mu \dot{x}_2(t) - \mu c x_2(t) - \mu c x_2(t) - \mu d u_2(t), \quad 0 \le t \le \sigma, \quad (14b)$$

where

$\alpha(t) - \hat{g}_{x_2}(t)$ and $\beta(t) - \hat{g}_{x_2}(t)$

are continuous functions on $[0,\sigma]$ and consider that

$x_i(.) \in D_n[0,\sigma]$ with $x_i(0) = 0 = x_i(\sigma)$, $i = 1, 2$; $n \ge 2$

then

$$\int_0^\sigma [x_1(t)(\alpha(t) - \hat{g}_{x_2}(t)) + \dot{x}_1(\beta(t) - \hat{g}_{x_2}(t))]dt = 0 \quad (15)$$

Applying a result due to Ref. 10 to Equation (14), it is not difficult to see that the problem of finding $g_{x2}(t)$ and $g_{u2}(t)$ is equivalent to that of solving the differential equation

$$\hat{g}_{x_2}(t) - \hat{g}_{x_2}(t) = \mu \ddot{x}_2(t) - \mu d u_2(t) - (a+\mu c^2)x_2(t) - \mu c d u_2(t). \quad (16)$$

But comparing equation (12) with (13) implies that

$$G_{21}g_{x_2} = \hat{g}_{u_2} = \mu c d x_2(t) + (b+\mu d^2)u_2(t) - \mu d \dot{x}_2(t) \quad (17)$$

By taking the Laplace transform of (16) and using the appropriate boundary conditions generated by (15), we obtain

$$\hat{g}_{x_2}(t) = -(\mu\dot{x}_2(0) - \mu c x_2(0) - \mu d u_2(0))\,sinh t + \int_0^t [\mu\dot{x}_2(s)$$

$$- \mu c x_2(s) - \mu c d u_2(s)\cosh(t-s)]ds + [(a+\mu c^2)x_2(0) + \mu c d u_2(0)$$

$$-\mu c\dot{x}_2(0)]\cosh t - \int_0^t [(a+\mu c^2)x_2(s) + \mu c d u_2(s) - \mu c x_2(s)]$$

$$\sinh(t-s)\,ds + \frac{Sinh t}{Sinh\sigma}\{(a+\mu c^2)x_2(\sigma) + \mu c d u_2(\sigma) - \mu c\dot{x}_2(\sigma)$$

$$+ (\mu\dot{x}_2(0) - \mu c x_2(0) - \mu d u_2(0))\,Sinh\sigma - \int_0^\sigma [\mu\dot{x}_2(s) - \mu c x_2(s) - \mu d u_2(s)]$$

$$\cosh(\sigma - s)\,ds + \int_0^\sigma [(a+\mu c^2)x_2(s) + \mu c d u_2(s) - \mu c\dot{x}_2(s)]\,Sinh(\sigma - s)\,ds \quad (18$$

$$- [(a+\mu c^2)x_2(0) + \mu c d u_2(0) - \mu c\dot{x}_2(0)]\cosh\sigma\}. \quad = G_{11} g_{x_2}(t).$$

Thus we have completely determined the Operators G_{11} and G_{21} as described by Equations (18) and (17) respectively.

Next, we construct the operators G_{12} and G_{22} by employing the same line of arguments as used in construction G_{11} and G_{21}. This time, we start by setting $g_{x2}(t) = 0$, and obtain

$$G_{22} g_{u_2}(t) = (b+\mu d^2)u_2(t) + \mu c d x_2(t) \quad (19)$$

and

$$G_{12} g_{u_2}(t) = \mu[c x_2(0) + d u_2(0)\,sinh t$$

$$- \mu\int_0^t [c x_2(s) + d u_2(s)]\cosh(t-s)\,ds - \mu\int_0^t [(a+\mu c^2)x_2(s)$$

$$+ \mu c d u_2(s)]\sinh(t-s)\,ds + [(a+\mu c^2)x_2(0) + \mu c d u_2(0)]\cosh t$$

$$+ \frac{Sinh t}{Sinh\sigma}\{(a+\mu c^2)x_2(\sigma) + \mu c d u_2(\sigma) - \mu(c x_2(0) - d u_2(0))\sinh\sigma$$

$$+ \mu\int_0^\sigma (c x_2(s) + d u_2(s))\cosh(\sigma - s)\,ds + \int_0^\sigma [(a+\mu c^2)x_2(s) +$$

$$+ \mu c d u_2(s)]\sinh(\sigma - s)\,ds - ((a+\mu c^2)x_2(0) + \mu c d u_2(0))\cosh\sigma\},$$

(20)

which fully completes the construction of operator G.

It is not difficult to see that when

$x_1(t) = x_2(t) = x(t)$

and

$u_1(t) = u_2(t) = u(t)$

then the operator G constructed above satisfies

$$<g, Gg> = \int_0^\sigma \{ ax^2(t) + bu^2(t) + \mu[\dot{x}(t) - cx(t) - du(t)]^2 \}dt$$

and we have

$$Gg = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} (G_{11}x)(t) + (G_{12}u)(t) \\ (G_{21}x)(t) + (G_{22}u)(t) \end{bmatrix} \qquad (21)$$

where

To apply the Operator G constructed here, in the Extended Conjugate Gradient Method (ECGM) algorithm, described in section two, we merely replace Ap_i appearing in Equation (8) by Gp_i. Here, Gp_i is computed as follows:

$$(Gp_i)(t) = \begin{bmatrix} (G_{11}p_{x,i})(t) + (G_{12}p_{u,i})(t) \\ (G_{21}p_{x,i})(t) + (G_{22}p_{u,i})(t) \end{bmatrix}, \quad 0 \le t \le \sigma$$

$$= \mu[2cp_{x,0} + 2du(0) - F_{x,0}]\sinh t + \mu\int_0^t [F_{x,i} - 2cp_{x,i} - 2du(s)]$$

$$\cosh(t-s)ds - \int_0^t [2(a+\mu c^2)p_{x,i} + 2\mu cdu(s) - \mu cF_{x,i}] \sinh(t-s)ds$$

$$+ [2(a+\mu c^2)p_{x,0} + 2\mu cdu(0) - \mu cF_{x,0}]\cosh t + \frac{\sinh t}{\sinh\sigma}\{2(a+\mu c^2)x(\sigma)$$

$$+ 2\mu cdu(\sigma) - \mu c\dot{x}(\sigma) + \mu(F_{x,0} - 2cp_{x,0} - 2du(0))\sinh\sigma - \mu\int_0^\sigma [F_{x,i} - 2cp_{x,i} - 2du(s)]$$

$$\cosh(\sigma-s)ds + \int_0^\sigma [2(a+\mu c^2)p_{x,i} + \mu c(2du(\sigma) - F_{x,i})]\sinh(\sigma-s)ds - [2(a+\mu c^2)p_{x,0}$$

$$+ 2\mu cdu(0) - \mu cF_{x,0}]\cosh\sigma\}, 2\mu dp_{x,i} + 2(b+\mu d^2)u(t) - \mu dF_{x,i},$$

(22)

where

$$F_{x,i} = F_x(x_i, u_i, \mu), \quad F_{u,i} = F_u(x_i, u_i, \mu)$$
$$p_{x,i} = p_x(x_i, u_i, \mu), \quad p_{u,i} = p_u(x_i, u_i, \mu)$$
$$F_i = F(x_i, u_i, \mu)$$

and

$$p_i = \begin{bmatrix} p_{x,i} \\ \\ p_{u,i} \end{bmatrix} = \begin{bmatrix} p_x(x_i, u_i, \mu) \\ \\ p_u(x_i, u_i, \mu) \end{bmatrix}$$

By making trivial modifications involving change of variables of the results given by Equation (21) we can determine the control operator G which satisfies

$$<g, Gg>_K = \int_0^\sigma (X^T(t)QX(t) + U^T(t)RU(t))dt + \mu\int_0^\sigma (\|\dot{X}(t) - CX(t) - DU(t)\|^2)dt, \quad \mu > 0$$

where as stated earlier,

$K = H_1[0, \sigma] \times L_2^n[0, \sigma]$ and $H_1[0, \sigma]$

denotes the Sobolev space of absolutely continuous functions X(.), square integrable with their first derivatives also square integrable over the closed interval [0,σ]. $L^n_2[0,\sigma]$ stands for the Hilbert space consisting of equivalence classes of square integrable functions from [0,σ] into IR^n, with norm denoted by $\|.\|$ and defined by

$$\|U\| = \{\int_0^\sigma \|U(t)\|_E^2 dt\}^{\frac{1}{2}}$$

and with scalar product conventionally denoted by <.,.> and defined by

$$<U_1, U_2> = \int_0^\sigma <U_1(t), U_2(t)>_E dt,$$

where $\|.\|_E$ and <.,.> denote the norm and inner product in the Euclidean n dimensional space. In Equation (23), g and G are as given by Equation (10), while X(t), U(t), Q,R C and D are as given by Equations (2) and (3).

The desired results given by Equations (24) - (27) below follow directly from the solution of similar second order ordinary differential equations with slight modifications of the analysis outlined in the construction of the control operator G above. Thus, we obtain the following more generalized results with respect to those given by Equations (17), (18), (19) and (20).

$$
\begin{aligned}
(G_{11}\dot{X})(t) &= -\mu(\dot{X}(0) - CX(0) - \mu DU(0))\sinh t + \mu\int_0^\sigma (\dot{X}(s) \\
&\quad - CX(s) - DU(s))\cosh(t-s)\,ds + [(Q + \mu C^T C)X(0) \\
&\quad - \mu C\dot{X}(0) + \mu C^T DU(0)]\cosh t - \int_0^t [(Q + \mu C^T C)X(s) \\
&\quad - \mu C^T DU(s) - \mu C\dot{X}(s)]\sinh(t-s)\,ds + \frac{\sinh t}{\sinh\sigma}[(Q + \mu C^T C)X(\sigma) \\
&\quad + \mu C^T DU(\sigma) - \mu C\dot{X}(\sigma) + \mu\sinh\sigma(\dot{X}(0) - CX(0) - DU(0)) - \mu\int_0^\sigma [\dot{X}(s) \\
&\quad - CX(s) - DU(s)]\cosh(\sigma-s)\,ds + \int_0^\sigma [(Q + \mu C^T C)X(s) + \mu C^T DU(s) \\
&\quad - \mu C\dot{X}(s)]\sinh(\sigma-s)\,ds - [(Q+\mu C^T C)X(0) + \mu C^T DU(0) - \mu C\dot{X}(0)]\cosh\sigma\,].
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
(G_{12}\dot{U})(t) &= \mu[CX(0) + DU(0)]\sinh t - \mu\int_0^t [CX(s) + DU(s)]\cosh(t-s)\,ds \\
&\quad - \int_0^t [(Q + \mu C^T C)X(s) + \mu C^T DU(s)]\sinh(t-s)\,ds + [(Q + \mu C^T C)X(0) + \mu C^T DU(0)] \\
&\quad \times\cosh t + \frac{\sinh t}{\sinh\sigma}\{(Q+\mu C^T C)X(\sigma) + \mu C^T DU(\sigma) - \mu[CX(0) + DU(0)]\sinh\sigma + \mu\int_0^\sigma [CX(s) + DU(s)] \\
&\quad \cosh(\sigma-s)\,ds + \int_0^\sigma [(Q+\mu C^T C)X(s) + \mu C^T DU(s)]\sinh(\sigma-s)\,ds - [(Q+\mu C^T C)X(0) \\
&\quad + \mu C^T DU(0)]\cosh\sigma\}.
\end{aligned}
\tag{25}
$$

$$(G_{21}\dot{X})(t) = \mu C^T DX(t) + (R + \mu D^T D(t) - \mu D\dot{X}(t), \tag{26}$$

and
$$(G_{22}\dot{U})(t) = RU(t) + \mu D^T DU(t) + \mu C^T DX(t), \tag{27}$$

from where (Gg)(t) can be computed as follows:

$$(Gg)(t) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \dot{X}(t) \\ \dot{U}(t) \end{bmatrix} = \begin{bmatrix} (G_{11}\dot{X})(t) + (G_{12}\dot{U})(t) \\ (G_{21}\dot{X})(t) + (G_{22}\dot{U})(t) \end{bmatrix} \quad ; \quad 0 \le t \le \sigma.$$

$(Gp_i)(t)$ is then computed as follows:

$$(Gp_i)(t) = \begin{bmatrix} (G_{11}p_{X,i})(t) + (G_{12}p_{U,i})(t) \\ (G_{21}p_{X,i})(t) + (G_{22}p_{U,i})(t) \end{bmatrix} \quad ; \quad 0 \le t \le \sigma, \tag{28}$$

where the right hand side of Equation (28) is obtained from Equations (24) to (27) as follows:

$$(G_{11}p_{x,i})(t) = (G_{11}\dot{X})(t)\big|_{\dot{X}_i = J_X(X_i,U_i,\mu);\ X_i = p_{X,i}},$$

$$(G_{12}p_{U,i})(t) = (G_{12}\dot{U})(t)$$

$$(G_{21}p_{X,i})(t) = (G_{21}\dot{X})(t)\big|_{\dot{X}_i = J_X(X_i,U_i,\mu);\ X_i = p_{X,i}},$$

$$\text{and} \quad (G_{22}p_{U,i})(t) = (G_{22}\dot{U})(t).$$

where $J(X,U,\mu)$ is given by:

$$J(X,U,\mu) = \int_0^\sigma [X^T(t)QX(t) + U^T(t)RU(t)]dt + \mu\int_0^\sigma [\| \dot{X}(t) - CX(t) - DU(t)\|^2]dt, \tag{29}$$

with X(t), U(t), Q, R, C, D and $\|.\|$ as given in Equation (23).

Next, we highlight the inherent properties of the control Operator G, constructed herein, which make the construction worthwhile.

The rate of convergence of the Conjugate Gradient method (CGM) is estimated using the knowledge of the eigenvalues of the Operator A, involved in the functional given by Equation (1). More appropriately, the rate of convergence of the Conjugate Gradient Method (CGM) is estimated using the knowledge of the least upper and greatest lower bounds of the Operator A, appearing in Equation (1). In order to obtain accurate or near accurate convergence estimates, the explicit determination of such eigenvalues of spectrum of the operator A is imperative.

It is then needless to say that any version of the Conjugate Gradient Method (CGM) that can solve the continuous optimal control Problem (1), requires first, the explicit determination of the control operator associated with the class of problems described by Equations (2) and (3).

Although, the construction of the Operator G, carried out in this paper is somewhat tedious, its needs very easily outweigh the difficulty involved in its construction. Furthermore, the explicit determination of the Operator G has thrown some green light on the convergence analysis of the Extended Conjugate Gradient Method (ECGM) originally proposed by the authors of Ref. 1 .
This point is addressed in Ref. 11.

4. Numerical Investigations:

The need for the construction of the operator G, undertaken in section 3 of this paper is confirmed by examples, as depicted in Tables 1, 2 and 3.

The superiority of the ECGM algorithm over the convensional function space algorithm of Ref. 5 had been confirmed in several reports, see Refs. 1, 6, 12 and 13. Our Examples (1), (2) and (3) had been solved in Ref. 14 via the ECGM algorithm using the operator A. Here, we employ the operator G, constructed herein and compare our results with the results obtained in Ref. 14. These examples permit a clear comparison of the effect of each of the operators on the ECGM algorithm.

For each of the examples, the optimal penalty constant μ^* is computed using the scheme proposed in Ref.14. Thereafter the values of

(i) $$OBJ = \int_0^\sigma [ax^2(t) + bu^2(t)]dt;$$

(ii) $$FUNCT = \int_0^\sigma \{ax^2(t) + bu^2(t) + \mu^*[\dot{x}(t) - cx(t) - du(t)]^2\}dt;$$

and

(iii) $$CSAT = \int_0^\sigma [\dot{x}(t) - cx(t) - du(t)]\,dt$$

are computed. The results obtained by using the operator A, of Refs. 2 and 4 are displayed under A in the respective tables, while those arising from the use of operator G, constructed herein are displayed under G.

Example (1). We wish to determine the control and state variables u and x respectively for which the cost functional

$$\int_0^1 (x^2(t)+u^2(t))\,dt$$

is minimized, subject to the dynamic constraint
$\dot{x}(t) = 2.095x(t) + 1.904u(t);\quad 0 \le t \le 1 \quad x(0) = 1.$

The solution has been found by applying the ECGM algorithm, given by (7) with the operators A and G used independently, with initial value $u_1(t) = 0.5$ and $\Delta t = 0.1$. The values for OBJ, FUNCT and CSAT as well as the number of iterations required to complete each minimization step are reported in Table 1.

Table 1.
Numerical Results For Example (1)

Iteration numbers	OBJ A	OBJ G	FUNCT A	FUNCT G	CSAT A	CSAT G
1	1.72484585	1.15270811	1.89779926	1.88618315	1.01341029	4.29775366
5	1.69185502	1.07133275	1.72436854	1.39477443	0.190211044	0.189518743
10	1.37396463	0.972324834	1.39602785	1.24135421	0.129278138	0.157636173

$\mu^* = 0.170664746$; Exact OBJ = 1.0647 [Ref. 6].

The theoretical solution (Exact OBJ) of Example (1) had been computed by the author of Ref. 6 as 1.0647. The application of the ECGM in solving the same problem using operator G yields and OBJ value of 1.07133275 after only five iterations, whereas the closest OBJ value obtained when the operator A is employed is 1.37396463 after nine iterations. As expected however, the FUNCT values approach the OBJ values as the iterations process progresses. The OBJ value of 1.37396463 points significantly to confirm the superiority of the ECGM over the conventional function space algorithm of Ref.5, which yields its nearest OBJ value of 5.7079 for the same problem.

Example (2): For a second computational example, we consider the minimization of the functional:

$$\int_0^5 (x^2(t) + u^2(t))\,dt$$

Subject to $\dot{x}(t) = x(t) + u(t)\ ;\ 0 \le t \le 5 \quad x(0) = 0.$

For the solutions given in Table 2, an initial value $u_{in}(t)$ of 0.5 was used, and $\Delta t = 1.0$.

Table 2.
Numerical Results For Example (2)

Iteration Number	OBJ A	OBJ G	FUNCT A	FUNCT G	CSAT A	CSAT G
0	1.25000	1.25000	1.87500	1.87500	1.2500	1.2500
1	1.00381039	0.421208535	1.48790056	0.724799961	0.968180343	0.60718285
5	0.587830958	0.316540317	0.743930506	0.519892466	0.312199096	0.40670429
10	0.317629755	0.3003807	0.3436345406	0.482344705	$5.74313 \times 10^{?}$	0.36392801

$\mu^* = 0.5.$

Whereas using operator A in the ECGM algorithm requires ten iterations to obtain an OBJ value of 0.317629755, the use of operator G yields a better OBJ value of 0.316540317, after only five iterations.

Example (3): The third and last example in this paper concerns the minimization of

$$\int_0^1 (x^2(t) + u^2(t))\,dt$$

subject to the dynamic constraint

$\dot{x}(t) = u(t)$, $0 \le t \le 1$ $x(0) = 1.0$.

Table 3. shows the numerical results for this example, taking an initial value $U_{in}(t)$ of 0.5 and $\Delta t = 0.1$.

Table 3

Numerical Solutions For Example (3)

Iteration Number	OBJ A	OBJ G	FUNCT A	FUNCT G	CSAT A	CSAT G
0	1.25000	1.25000	1.5000	1.5000	0.2500	0.2500
2	1.2145917	1.06385009	1.28042803	1.0708152	6.58373×10^{-2}	6.95144×10^{-3}
5	1.0620548	0.978850422	1.12274958	0.98205674	6.06948×10^{-2}	4.00632×10^{-3}
10	0.96476564	0.951115584	1.02446509	0.954700967	5.96994×10^{-3}	3.58538×10^{-3}

$\mu^* = 1.0$

5. CONCLUSIONS

Our results depicted in Tables 1,2 and 3 are completely in line with the principle of constrained Optimization that: when a constrained problem is solved via an unconstrained scheme, it is hoped that the solution of the unconstrained problem will eventually tend to the solution of the constrained problem. Here the solution of the constrained problem is denoted by OBJ, while that of the unconstrained problem is represented by FUNCT. The constraint satisfaction is denoted by CSAT.

For each of the Examples, we observe from Tables 1,2 and

3, that the FUNCT values approach the OBJ values as the iterations process progresses.

The idea here is not to proselytize by intimidating the reader with the three examples demonstrating the full scope of the technique in question, but rather to present to the reader careful treatment of just one family of control problems, namely, the one-dimensional case, (for simplicity, as there is no loss of generality) to which our analysis can be successfully applied. On taking note of the various contributions to knowledge documented in the literature, we believe that an extension of our numerical results to embrace optimal control problems governed by systems of dynamic constraints is quite conceivable, having presented the expressions for the theoretical result in Equations (24), (25), (26) and (27).

Meanwhile, it is modest to conclude this paper with the following note. While the various modifications of the Conjugate Gradient method are interesting on their own rights, it is hoped that the exposition in this paper will also serve to indicate the advantages and capabilities of the ECGM algorithm with respect to the conventional penalty functions space methods for solving regulator problems. We prefer to substantiate this claim with Table 1 which concerns Example (1).

From Table 1, it is seen that the ECGM algorithm with Operator A used, yields the closest objective functional value of 1.37396463 after nine iterations, while it yields an objective functional value of 1.07133275 after five iterations, when the Operator G, constructed in this paper is used. It is of interest to note here; that the Exact (Theoretical) objective functional value of the problem (Example (1)) is 1.0647; that the conventional function space algorithm of Ref. 5 yields its closest objective functional value of 5.7079 after fifty iterations. Thus the use of operator G in the ECGM algorithm not only yields a near accurate objective functional value, it minimizes computing times (when compared with the use of operator A) via the minimization of number of iterations required to obtain a close approximation to the theoretical solution. We have equated computing times to number of iterations here because the number of function evaluations is the same for the operators A and G.

In all, if our overall objective in solving our class of problems, is to keep 'computing times' as low as possible, while obtaining very good approximations to the objective functional values via the penalty function method - ECGM; then the use of Operator G is imperative.

REFERENCES

1. Ibiejugba, M.A. and Onumanyi, P. "On a Control Operator and some of its applications". Journal of Math Anal. and Applications Vol. 103, pp. 31 - 47 1984.
2 Ibiejugba, M.A. "Computing Methods in Optimal Controls", Ph.D Thesis, Univ. of Leeds. 1980.
3. Hestens, M.R. and Stiefel, E.; "Method of Conjugate Gradients for solving linear systems". Journal of Research of the National Bureau of Standards, Vol. 49, pp 409 - 436, 1952.
4. Ibiejugba, M.A."The Construction of a Control Operator" An Stiint Univ. Al. I. Cuza' XXVIII, pp. 45 - 48 1982.
5. Di Pillo, G.; Grippo, L. and Lampariello, F; "The Multiplier Method for Optimal Control Problems", Conference on Optimization Engineering and Economics, Naples, Italy, 1974.
6. Aderibigbe, F.M. "An Extended Conjugate Gradient Method

for Evolution Equations". Ph. D Thesis, Univ. of Ilorin, Nigeria 1987.

7. Ibiejugba, M.A., and Buraimoh-Igbo, L.A. "A Monotonic Convergence property of the Extended Conjugate Gradient Method Algorithm". Advances in Modelling and Simulation, Vol. 2, pp. 1-9 1985.

8. Liusternik, L. A. and Sobolev, V. I.; "Elements of Functional Analysis": Frederick Ungar. English Translation. New York. 1961.

9. Luenberger, D. G.: "Optimization by Vector space Method": John Wiley & Sons, Inc. London 1969.

10. Curtain, R. F. and Pirtchard, A. J."Functional Analysis in Modern Applied Mathematics, Academic Press, London. 1977

11. Otunta, F. O. "A Convergence Estimate for the ECGM" Abacus Journal of MAN, Vol. 24, NO. 2,1994

12. Ibiejugba, M. A. and Rubio, J. E." A Penalty Optimization Technique for a class of Regulator problems"; JOTA, Vol. 58, NO. 1. pp 39-62. 1988.

13. Ibiejugba, M.A. and Aderibigbe, F.M. "A penalty Optimization Technique for a class of regulator problems part 4". Abacus J. Math. Association of Nigeria. (In Press) 1991.

14. Otunta, F.O. "Optimization Techniques for a Class of Regulator Problems". Ph.D. Thesis, University of Ilorin Nigeria 1991.