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ON A NONLINEAR DIFFERENTIAL EQUATION FOR THE M(t)|D|1 QUEUEING SYSTEM

UWAMUSI, STEPHEN EHIDIAHEN
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
UNIVERSITY OF BENIN, BENIN CITY, NIGERIA

ABSTRACT

A simple queueing characterized by a time dependent distributed arrival rate and a discrete service time distribution formulated as a nonlinear first order differential equation in line with Rider [7] is considered. We modified the inter-arrival time distribution for the queue size using some techniques of trigonometric sine wave. This enables us to have cyclic behaviour of expected queue size as well as the probability that the server is idle. We realized our numerical solution thereof using the well known approach of **predictor-corrector pairs evaluate mode (PEC)^m**. Our values obtained compare favourably with those of Rider's M(t)/m(t)/1.

1 INTRODUCTION

We define a queue as a line of people, cars machines e.t.c waiting for (some kinds of attention) service. We also refer to item waiting for service as a customer irrespective of its .nature. Thus a complete queueing system includes the customers queueing up for service (waiting line) and a service facility.

In line with Cox and Fox [1] it is given that one of the notation for specifying a queueing system in which a customer may require some service(s) could be represented by Kendall's notation a/b/s. Here a represents the inter-arrival time distribution, b represents service time distribution and s is the number of service.

It is usual to modify Kendall's notation. The modified version of this is A(t)/B(t)/1. The (t) is to indicate that the parameters of the distribution is a function of t while 1 indicates that there is just only one server in the system.

Our motivation into this research is purely on the treatments of time dependent queueing system. The remaining sessions in this paper are categorized as follows:

Section two deals in general approach formulation of differential equation for the queue system based on the Chapman-Kolmogorov's differential equation by carefully adopting the approach of Rider [7].

In section three, the integrator of modified Euler which serves a basis of Linear Multistep method was applied to our method to provide numerical solution.

2 MAIN RESULT – GENERAL APPROACH.

Following closely the idea of Rider [7] as well as [3,4,6, and 8] a simple differential equation for queue size could be obtained. A basic fundamental of this approach could be found in Feller [2].

In the sequel, the following abbreviations shall be noted.

$P_j(t)$ = Probability that there are j jobs in the system at time t,

$\lambda(t)$ = arrival rate of jobs at time t,

$\mu(t)$ = service rate of jobs at time t,

$\rho(t) = \lambda(t)/\mu(t)$ = traffic intensity

We consider $P_j(t)$ and $P_j(t+h)$. When $j > 0$ the probability that there will be j jobs at time $(t+h)$ is expressed as the sum of three independent compound probabilities.

(i) the probability that there are j jobs in the system at time t, and no arrivals and no departures during h. This probability is $P_j(t)(1-\lambda(t)h)(1-\mu(t)h)$

(ii) the probability that there are j+1 jobs in the system at time t and one departure and no arrival during h. This probability is $P_{j+1}(t)(1-\lambda(t)h)\mu(t)h$.

(iii) the probability that there are j-1 jobs in the system at time t and no departure and one arrival during h. This probability is $P_{j-1}(t)(1-\mu(t)h)\lambda(t)h$.

We use the relative slowness and smoothness of probability on the short interval h as justification for replacing such increments as $P_j(t+h) - P_j(t)$ by the linear approximation, $P_j'(t)$. The events that will occur with probability containing terms in h^2 and higher powers of h are ignored. Considering conditions (i) –(iii) we have the differential equations for the queue size

$$P_j'(t) = \lambda(t)P_{j-1}(t) - (\lambda(t) + \mu(t))P_j(t) + \mu(t)P_{j+1}(t) \quad (2.1)$$

For j = 0, we have

$$P_o(t) = -\lambda(t)P_o(t) + \mu(t)P_1(t) \quad (2.2)$$

Denote

$$\psi(t) = \sum_{j=1}^{\infty} jP_j(t) \quad (2.3)$$

as the expected queue size

Differentiation of (2.3) yields

$$\psi'(t) = \sum_{j=1}^{\infty} jP'_j(t) \tag{2.4}$$

Thus (2.4) in conjunction with (2.1) we have

$$\psi'(t) = \sum_{j=1}^{\infty} j(\lambda(t) + P_{j-1}(t) - (\lambda(t)) + \mu(t)P_j(t) + \mu(t)P_{j+1}(t)) \tag{2.5}$$

Summing term by term, we have

$$\psi'(t) = \lambda(t)\sum_{j=1}^{\infty} jP_{j-1}(t) - (\lambda(t) + \mu(t))\sum_{j=1}^{\infty} jP_j(t) + \mu(t)\sum_{j=1}^{\infty} jP_{j+1}(t)$$

that is

$$\psi'(t) = \lambda(t)P_o(t) + \sum_{j=2}^{\infty} jP_{j-1}(t) - (\lambda(t) + \mu(t)\psi(t) + \mu(t)\sum_{j=1}^{\infty} jP_{j+1}(t)$$

At this point some mathematical logic comes into play by adopting the following facts

We set r = j-1 $\Rightarrow$ j = r + 1

Therefore $\sum_{j=2}^{\infty} jP_{j-1}(t) = \sum_{j=1}^{\infty} (r+1)P_r(t) = \sum jP_j(t) + (1 - P_O(t))$

Thus

$$\lambda(t)\sum_{j=1}^{\infty} jP_{j-1}(t) = \lambda(t)(P_o(t) + \psi(t) + 1 - P_o(t))$$

$$= \lambda(t) + \lambda(t)\psi(t) = \lambda(t)(1 + \psi(t))$$

Also $\sum_{j=1}^{\infty} jP_j(t)(\lambda(t) + \mu(t))$

$$= \psi(t)(\lambda(t) + \mu(t))$$

As for the term $\sum_{j=1}^{\infty} jP_{j+1}(t)$, we have that j + 1 = r $\Rightarrow$ j = r-1

Thus $\sum_{j=}^{\infty} jP_{j+1}(t) = \sum_{j=2}^{\infty} (r-1)P_r(t)$

$$= \psi(t) - \sum_{j=1}^{\infty} P_j(t) = \psi(t)(1 - P_o(t))$$

It follows that

$$\begin{aligned}\psi'(t) &= \lambda(t)(P_o(t)+\psi(t)+1-P_o(t)) \\ &-(\lambda(t)+\mu(t))\psi(t)+\mu(t)(\psi(t)-1-P_o(t)) \\ &= \lambda(t)-(\mu(t)+\mu(t)P_o(t)) \\ \text{i.e. } \psi'(t) &= \lambda(t)-\mu(t)(1-P_0(t))\end{aligned} \qquad (2.5)$$

Observe that equation (2.5) is a non linear first order differential equation involving $\lambda(t)$, $\mu(t)$ and the probability of idle server, $P_o(t)$.

Remark that it is extremely difficult to integrate equation (25) owing to the term $P_o(t)$ relating to $\lambda(t)$ and $\mu(t)$. In [7], the expression of P_o is given in the form

$$P_0(t) \equiv [1-\alpha(t,T)][1-\rho(t)] + \frac{\alpha(t,T)}{(1+\psi(t))}$$

where

$$\sigma(t,T) = e^{-\mu(t)T} \qquad 2.6$$

and T is the probability relaxation shape parameter time

$$\psi'(t) = \lambda(t) - \mu(t)\left(1-\left\{\left(1-\bar{e}^{\mu(t)T}\right)(1-\rho(t)) + \frac{e^{-\mu(t)T}}{(1-\psi(t))}\right\}\right. \qquad (2.7)$$

Expanding (2.7) and using the fact that $\rho(t) = \lambda(t)/\mu(t)$. We obtain expression for $\psi'(t)$ if we multiply and divide by $\mu(t)$ thus:

$$(\psi')t = \mu(t)e^{-\mu(t)T}[\rho(t)-\sigma(t)] \qquad (2.8)$$

$$Where\ \sigma(t) = \frac{\psi(t)}{(1+\psi(t))}$$

No detail was given by Rider about the numerical method adopted in equation (2.8). In the next section we shall discuss the numerical method we wish to integrate equation(2.8).

3. LINEAR MULTISTEP METHOD APPROACH

We set to adopt the predictor corrector pair of modified Euler via Linear multistep method (LMM). We define LMM as follows:

$$\mathcal{L}(\psi(t),h) = \sum_{j=0}^{K}\alpha_j\psi_{n+j} - h\sum_{j=0}^{K}B_j f_{n+j} \qquad (3.1)$$

where $f_{n+j} = f(t_{n+j}, \psi_{n+j})$ and ψ_{n+j} is called the theoretical solution of $\psi(t_{n+j})$ taken at the point $t = t_{n+j}$. k is the step number and h is the usual step size. We say that LMM is explicit if $B_k = 0$ otherwise it is implicit. Thus if $B_k =$

0, LMM is predictor while if $B_k \neq O$ LMM is corrector. It is usual to normalize LMM and its normalization is of the form

$$\psi_{n+k} + \sum_{j=O}^{K-1} \alpha_j \psi_{n+j} = h \sum_{j=0}^{K} B_j f_{n+j} \qquad (3.2)$$

Denoting P as the predictor and C as the corrector, then the predictor - corrector pair evaluate mode $(PEC)^m$ is

$$P : \psi^{(S)}_{n+k} = h \sum_{j=O}^{k-1} B_j \bar{f}_{n+j} - \sum_{j=O}^{k-1} \alpha_j \bar{\psi}_{n+j}$$

$$C : \psi^{s+1}_{n+k} + \sum_{j=O}^{k-1} \psi^{(s)}_{n+j} = h[B_k f(t_{n+k}), \psi^{(S)}_{n+j} + h \sum_{j=O}^{k-1} B_j f_{n+j}]$$

(s = 0, 1, ...) (3.3)

It can be shown that the order 0 of a LMM is I given by

$$\mathcal{L}(\psi(t), h) = \sum_{j=1}^{K} \left[\frac{j^p \alpha_j}{j!} \quad \frac{j^{p-1} B_j}{(p-1)!,} \right]$$

The proof of these can be found in computational method in ordinary differential equation by Lambert [5]. In this paper our emphasis has been made to study M(t)/D/ 1 system where D is the deterministic service holding time using five unit. Time is optional in any case.

In particular the inter arrival time distribution has a trigonometric sin wave given as

$$\lambda(t) = 4 \sin\left(\frac{At}{24}\right), \qquad (t = 1, 2, \ldots 24) \qquad (3.4)$$

This enables us to have a cyclic behaviour of queue with cyclic inter arrival rate. Results for the measures of performance-expected queues size and probability of idle server are displayed in table 1. The values of T correspond to inverse function of service holding time in the interval [0.4, 1]. For t = 1(1) 48 in (3.4) leads to cyclic effect of our result.

4 CONCLUSION

A system of M(t)|D|1 was constructed in line with Rider (7). A modification of the inter - arrival time distribution as a trigonometric sine wave was introduced. Furthermore, we allowed the service rate to assume a constant holding time of five units. In this way, we have constructed a cyclic M(t)|D|1 system. An interval of [0.4,1] was chosen and a complete measure of total variation of this interval was made. The ratio of this total variation to the constant service holding time of five units gave rise to what is termed the probability distribution shape relaxation time which is very crucial factor for the approximate queue size Q′(t) that is dependent on the probability of idle server, $P_o(t)$.

Table 1 is the set of values we obtained using the predictor – corrector pair evaluate $(PEC)^m$ to our equation (2.8). Table 2 is the set of results obtained by Rider (7). A cursory look shows that our results in Table 1 are not too far away from that of Table 2.

Our model, M (t)|D|1, may be found applicable in studying a chaotic situation where there is a rush hour time. It may also be useful in the analysis of Forke – Plank equation. An extension of this model M(t)|D|1, system to include system of non-linear equations is recommended for future researchers.

Table 2 Result obtained by Rider [7]

Time Period	DATA		EXPECTED QUEUE SIZE			PROBABILITY OF IDLE SERVICE		
	Arrival rate Value	Arrival rate Value	Exalt Value	New Approx. T=0	New Approx. $T=\frac{.4}{\mu(t)}$	Exact Value	New Approx. T=0	New Approx. $T=\frac{.4}{\mu(t)}$
1.	12.0	15.0	4.52	4.95	4.65	0.18	0.17	0.18
2	10.0	13.0	6.63	3.72	3.86	0.25	0.21	0.21
3	7.0	13.0	2.66	2.35	2.80	0.34	0.30	0.33
4	52.0	13.0	1.29	0.81	1.15	0.53	0.55	0.51
5	4.5	7.0	0.69	0.55	0.58	0.63	0.65	0.54
6	4.5	7.0	1.39	1.43	1.28	0.40	0.41	0.41
7	5.0	10.0	1.74	1.84	1.67	0.34	0.35	0.42
8	5.5	10.0	1.28	1.24	1.28	0.45	0.45	0.44
9	6.5	13.0	1.49	1.52	1.46	0.39	0.40	0.44
10	7.0	13.0	1.17	1.14	1.16	0.46	0.47	0.46
11	7.0	15.0	1.1	1.16	1.16	0.46	0.46	0.49
12	7.0	15.0	0.90	0.88	0.89	0.53	0.53	0.53
13	7.0	15.0	0.88	0.88	0.88	0.53	0.53	0.53
14	7.0	15.0	0.88	0.88	0.88	0.53	0.53	0.53
15	7.0	88.0	0.88	0.88	0.88	0.53	0.53	0.40
16	7.0	8.0	2.40	2.39	2.27	0.24	0.27	0.25
17	7.0	12.0	3.14	3.61	3.05	0.20	0.22	0.30
18	10.0	12.0	2.81	3.07	2.84	0.24	0.25	0.23
19	10.0	12.0	3.41	3.77	3.44	0.20	0.21	0.21
20	10.0	15.0	3.77	4.17	3.83	0.19	0.19	0.25
21	10.0	15.0	2.70	2.68	2.83	0.29	0.27	0.28
22	13.0	15.0	3.18	3.20	3.12	0.22	0.1-	0.21
23	13.0	15.0	4.03	4.32	3.98	0.17		
24	13.0	15.0	4.53	497	4.54	0.16		

Time period	Arrival rate	Service rate							PROBABILITY OF IDLE SERVER						
.			.4/ mμ(t)	.5/ mμ	.6/ mμ	.7 mμ	.8 mμ	.9/m μ	.1/ mμ	.4/ mμ	.5/ mμ	.6/ mμ	.7/mμ	.8/m μ	.9/ mμ	1/ mμ
1	0.26	5.00	4.27	4.29	4.31	4.33	4.348	4.37	4.38	0.44	0.49	0.53	0.57	0.61	0.64	0.67
2	0.52	5.00.	3.40	3.43	3.45	3.46	3.48	-3.49	3.51	0.45	0.49	0.53	0.56	0.59	0.62	0.65
3	0.78	5.00	2.47	2.49	2.51	2.52	2.53	2.55	2.56	0.47	0.51	0.54	0.57	0.59	0.62	0.64
4	1.04	5.00	1.18	1.19	1.20	1.21	1.21	1.22	1.23	0.57	0.59	0.61	0.62	0.64	0.65	0.37
5	1.29	5.00	0.64	0.65	0.65	0.66	0.66	0.66	0.66	0.65	0.66	0.67	0.67	0.68	0.68	0.39
6	1.55	5.00	1.30	1.31	1.32	1.33	1.33	1.34	1.34	0.52	0.53	0.55	0.56	0.57	0.58	0.59
7	1.77	5.00	1.65	1.66	1.37	1.67	1.68	1.68	1.69	0.47	0.48	0.50	0.51	0.52	0.52	0.55
8	2.00	5.00	1.22	1.23	1.24	1.24	1.25	1.25	1.25	0.50	0.51	0.52	0.52	0.53	0.52	0.54
9	2.22	5.00	1.44	4.45	1.45	1.45	1.46	1.46	1.46	0.46	0.47	0.47	0.48	0.49	0.50	0.50
10	2.44	5.00.	1.15	1.16	1.16	1.16	1.16	1.16	1.16	0.48	0.48	0.49	0.49	0.49	0.49	0.49
11	2.64	5.00	1.17	1.17	1.17	1.17	1.17	1.17	1.17	0.47	0.47	0.48	0.47	0.47	0.47	0.47
12	2.83	5.00	0.93	0.93	0.92	0.92	0.92	0.92	0.92	0.49	0.49	0.47	0.48	0.47	0.17	0.47
13	3.01	5.00	0.92	0.92	0.91	0.91	0.91	0.91	0.90	0.48	0.47	0.45	0.46	0.45	0.45	0.44
14	3.17	5.00	0.93	0.93	0.92	0.92	0.91	0.90	0.91	0.47	0.46	0.43	0.44	0.44	0.43	0.42
15	3.33	5.00	0.94	0.94	0.93	0.93	0.92	0.92	0.91	0.46	0.44	0.30	0.43	0.42	0.41	0.40
16	3.46	5.00	2.40	2.40	2.40	2.40	2.40	2.40	2.40	0.30	0.30	0.26	0.30	0.30	0.30	0.30
17	3.59	5.00	3.13	3.13	3.13	3.13	3.13	3.13	3.13	0.26	0.26	0.26	0.26	0.26	0.26	0.27
18	3.69	5.00.	2.81	2.81	2.81	2.81	2.81	2.81	2.81	0.26	0.26	0.23	0.26	0.26	0.26	0.26
19	3.79	5.00	3.41	3.41	6.41	3.11	3.41	3.41	3.41	0.23	0.23	0.22	0.23	0.24	0.24	0.24
20	3.36	5.00	3.76	3.76	377	3.77	3.77	3.77	3.77	0.22	0.22	0.25	0.22	0.22	0.22	0.22
21	3.92	5.00	2.72	2.72	2.71	2.71	2.71	2.71	2.71	0.25	0.25	0.25	0.24	0.24	0.24	0.24
22	3.97	5.00	3.19	3.19	3.19	3.19	3.19	3.19	3.19	0.23	0.22	0.22	0.22	0.22	0.22	0.22
23	3.99	5.00	4.03	4.03	4.03	4.03	4.03	4.03	4.03	0.20	0.20	0.20	0.20	0.20	0.20	0.20
24	4.0	5.00	4.52	4.52	4.52	4.52	4.52	4.53	4.53	0.19	0.19	0.19	0.19	0.19	0.19	0.19

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