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FURTHER ON THE DAMPING OF LOW FREQUENCY SEISMIC WAVES IN A SEMI-INFINITE ELASTIC MEDIUM

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SUMMARY:

A damping term with coefficient γ is incorporated into the equations governing the evolution of Rayleigh waves in a semi-infinite elastic medium. The geophysical interests in this approach are examined.

In this consideration, it is shown that γ satisfies a quartic equation. The condition that the four roots are real is derived and verified. Therefrom, a model of an elastic medium which satisfies the Poisson's relation is considered and the dependence of the decay term for the complex phase velocity C on γ is generalised. Therefrom, a number of interesting results emerge and discussed.

1 INTRODUCTION

The excitation and propagation of Rayleigh surface waves may be analysed with the aid of the associated function derived from the equations governing the wave process. In an undamped medium, the function is usually a cubic polynomial in the powers of $(C_o/\beta)^2$. Two different versions of this function had been considered[2,4]. In the formulation, C_o and β are respectively, the phase speed of the surface wave and shear wave speed.

The elastic surface waves are generally non-dispersive; however, C_o dependents on the Poisson's constant ν for the medium and β is the entire function of the material rigidity. Further, taken α as the corresponding velocity of the associated compressional wave velocity, the Poisson's relation gives $\alpha = \beta\sqrt{3}$. This relation applies to most elastic materials and simplifies analytical study therefrom. With this simplification, Rayleigh function factorises with three distinct real roots but only one of them satisfies the inequality $C_o < \beta < \alpha$. This root is identified with the non-dispersive surface wave and thus, in agreement with experimental results.

In this paper, we consider the phenomena of small amplitude damped Rayleigh surface wave in a material. In the case of such wave system, the damping term is usually small. Correspondingly, the effects of the damping term in the Rayleigh function and associated propagation of the Rayleigh wave are analysed in this paper.

Following previous work in this topic[3,2,8], our model of elastic solid is characterised by a damping term which is proportional to the time rate of change of the material displacement components. The damping term seems to have incorporated the effects of the material inherent irregularities to some extent. However, in the case of finite amplitude surface waves it is usually preferable to introduce random terms in the material density and rigidity.

An identical model of elastic wave with damping term but in this case, proportional to the velocity failed to give results in reasonable agreement with experiment[5]. The present model, therefore suggests an improvement. This is clearly seen when the model is applied in the study of oscillations with small amplitude propagating near the earth's surface.

The area of geodynamics where the present model usually achieves a measure of success is in the theory of microseisms induced by the random pressure variation associated with the shallow water waves. The amplitude variations of these micro-scale signals were computed using the present model. In this case, the agreement with the observed data was quite close within a wide range of microseisms frequencies[3,7]. Hence, the motivation in the use of this model in the present investigation looks encouraging.

2. THE GOVERNING EQUATION AND THEIR SPECIFICATIONS.

In the two dimensional model x - axis is horizontal and normal to the wave front. z-axis points vertical downwards with $z = 0$ as the seabed. Thus, the model is on the negative side of the z-axis. $\phi(x,z,t)$ and $\Psi(x, z, t)$ are the scalar potentials associated with the material solid. ρ is the corresponding material density. Consequently, the compressional wave speed α and shear wave speed β are given by:

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho} \tag{2.1}$$

$$\beta^2 = \mu/\rho \tag{2.2}$$

Introducing a damping term in the system of equations governing the oscillations in an elastic half space[2,3], we have:

$$P(k)\exp ik(x-ct) = \lambda\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right) + 2\mu\left(\frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\psi}{\partial x\partial z}\right) + \gamma\rho\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial z}\right) \tag{2.3}$$

$$\mu\left(2\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \gamma\rho\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial z} - \frac{\partial\psi}{\partial x}\right) = 0 \tag{2.4a}$$

The displacement components (u,w) are given in term of ϕ and Ψ by the following representations:

$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial z} \tag{2.4b}$$

$$w = \frac{\partial\phi}{\partial z} - \frac{\partial\psi}{\partial x} \tag{2.4c}$$

γ is the damping coefficient, the random pressure amplitude associated with the generating source is given by P(k). c on the left hand side of (2.3) is the speed of the high phase velocity component of the water waves. It matches that of the seismic surface wave initiated by P(k) and this ensures the resonant transfer of energy from surface gravity waves to underlying elastic bottom motions.

2.3 and 2.4(a) are satisfied by the following relations:

$$\phi = A\exp[ik(rz + x - ct)] \tag{2.5}$$

$$\Psi = B\exp[ik(sz + x - ct)] \tag{2.6}$$

where A and B are constants, k = wave number for the medium.

$$r^2 = \left(\frac{c^2}{\alpha^2} - 1\right), s^2 = \left(\frac{c^2}{\beta^2} - 1\right) \tag{2.7}$$

Since 2.3 and 2.4(a) contain a damping term respectively, 2.5 and 2.6 represent damped sinusoidal oscillations. As a result, c is complex and has the representation $C = c_o + i\delta c$, $\delta c < c_0$.

Consider the far field behavior of 2.3 to 2.6 in which the oscillations are free from the generating source, the following relations apply:

$$A\,[\beta^2(s^2-1) + \gamma C] + B[2\beta^2 s - \gamma CS] = 0 \qquad (2.8)$$

$$A\,[2r\beta^2 - \gamma Cs] + B[\beta^2\,(1 - s^2) - \gamma C] \qquad (2.9)$$

3. THE RAYLEIGH'S FUNCTION f(c).

The consistency equation for 2.8 and 2.9 gives the Rayleigh function as follows:

$$f(c) = (2\beta^2 - \gamma c)^4\, rs + [\beta^2(1 - s^2) - \gamma c]^4 = 0 \qquad (3.1)$$

Using 2.7, 3.1 simplifies to $f(c/\beta)$ where

$$f(c/\beta) = (2\beta^2 - \gamma c)^4 \left(\frac{\alpha_1^2 C^2}{\beta^2} - 1 \right) - \left[\beta^2(2 - c/\beta) - \gamma c\right]^4 = 0 \qquad (3.2)$$

and

$$\alpha_1^2 = \beta^2/\alpha^2 = \frac{1-2\nu}{2-2\nu},\ \nu = \frac{\lambda}{2(\lambda+\mu)} \qquad (3.3)$$

Since $\lambda > 0$, $\mu > 0$ then, $\nu < 1$ and α_1^2 is positive. Usually, $\nu \varepsilon [-1, \tfrac{1}{2}]$.

Let $k_1 = c/\beta$: if $f(k_1) = 0$ then, 3.2 takes the form:

$$(2\beta - \gamma k_1)^4 (\alpha_1^2 k_1^2 - 1) - [\beta(2 - k_1) - \gamma k_1]^4 = 0 \qquad (3.4)$$

Re-arranging 3.4 in powers of k_1, then

$$k_1^6(\beta^4 - \alpha_1^2\gamma^4) + 4\gamma\beta k_1^5(\beta^2 + 2\gamma^2\alpha_1^2) - k_1^4[8\beta^4 - \gamma^2\{6\beta^2(4\alpha_1^2 - 1)$$

$$- \gamma^2(\alpha_1^2 + 1)\}] + 4\beta\gamma k_1^3[\gamma^2(2\alpha_1^2 + 1) + 2\beta^2(4\alpha_1^2 - 3)] + k_1^2 \qquad (3.5)$$

$$[8\beta^4(3 - 2\alpha_1^2) + 24\beta^2\gamma^2\alpha_1^2] + 16\beta^3\gamma k_1(1 - 2\alpha_1^2) - 16\beta^4(1 - \alpha_1^2) = 0$$

Equation 3.4 looks interesting because (i) it is no longer a cubic in k_1^2 with the introduction of γ; (ii) if $\gamma = 0$, we obtain the usual Rayleigh function for an undamped elastic medium; (iii) k_1 depends on μ, λ and γ, but not on the frequency of the excited system, thus, the propagation is non-dispersive. The statement (iii) seems to have confirmed the observed non-dispersive behavior of Rayleigh waves and their association with the generating non-dispersive shallow water waves. In the later case, the phase speed c_ω depends only on the water depth h_o below the surface of the shallow water; that is $c_\omega^2 = gh_o$, g = acceleration of gravity.

4. ANALYSIS OF EQUATION (3.5)

$f(0) = -16\beta^4(1 - \alpha_1^2) < 0$, since $\alpha_1^2 = \beta^2/\alpha^2 < ½$ for most of the elastic materials,
$f(1) > 0$ corresponds to:

$$\gamma^3[\alpha_1^2(2\gamma-4\beta)+(\gamma-4\beta)]+6\beta^2\gamma^2[1-\alpha_1^2]+\beta^3(4\gamma-\beta)<0$$

because each term in the bracket is negative if $\gamma < \beta/4$. This is reasonable since γ is usually small in a linear system. Thus, there is, at least, a root of (3.5) lying between 0 and 1 for all realistic values of α_1^2.
For $f(-1)$, we have

$$\beta^4+4\gamma\beta^3+6\beta^2\gamma^2(8\alpha_1^2-1)-4\beta\gamma^3(4\alpha_1^2+1)-\gamma^4(2\alpha_1^2+1)$$

$$>\frac{10}{16}\beta^4+59\frac{7}{8}\beta^2\alpha_1^2Y^2+3\frac{31}{32}\beta^3\gamma>0, if\ \gamma<\beta/4.$$

Thus at least, four complex conjugate roots of (3.5) lie within the circle $|k_1| < 1$; one in each quadrant of k_1 - plane.

Further, (3.5) contains terms involving γ, even and odd powers of k_1. Unlike the counterpart equation for perfect elastic solid, it does not reduce to a cubic in k_1^2 that can be readily analysed by the use of existing theory. Consequently, to examine the nature of zeros of (3.5) in $|k_1| < 1$, sequence $\{f_m(k_1)\}$, m = 0, 1, ..., 5 of Sturm's functions are computed for the problems[6]. In this consideration, $f_0(k_1) = f(k_1)$ which is equation (3.5); $f_1(k_1)$ is the derivative of $f(k_1)$ and so on. Let c(0) be a number assigned to the changes of sign in these sequences when $k_1 = 0$ and let a similar meaning be attached to c(1) when $k_1 = 1$. We find that the difference c(0) - c(1) in $0 < k_1 < 1$ depends on the range of values of γ.

If $0 \le \gamma \le \alpha_1^2\beta/40$, then c(0) - c(1) = 1 and there is only one real root in the interval $0 \le k_1 \le 1$. In this case, the effect of damping is negligible in the propagation of elastic waves. With regards to the Sturm's sequence, all the leading coefficients are positive. If $\alpha_1^2\beta/40 < \gamma \le \beta/4$, the leading coefficients for m = 2 and m = 3 in the Sturm's functions are negative and c(0) - c(1) = 2. Thus, in $0 \le k_1 \le 1$, there are two complex conjugate roots for (3.5) in the k_1-plane. And, this is indeed the sufficient condition for damped propagation. In particular, the complex root for which $\text{Re}\{k_1\} > 0$, $\text{Im}\{k_1\} > 0$ corresponds to the damped waves.

Considering the microseismic signals in the spectrum of surface waves, the phase velocity ranges from 1.6 km/sec to 2.6 km/sec and the corresponding γ ranges from 0.024km/sec to 0.054km/sec. Therefore, the value of γ is within the range $\alpha_1^2\beta/40 < \gamma < \beta/4$ and suggests that microseismic signals are damped; given that $\beta = 2.75$km/sec and a distance decrement of 1.31 H 10^{-3}/km.

5. The damping - γ and decay- δc factor.

From 3.5, γ satisfies the quartic equation

$$(\gamma k_1)^4(\alpha_1^2+1)-4\beta(\gamma k_1)^3(2\alpha_1^2k_1^2+2\alpha_1^2+1)-6\beta^2(\gamma k_1)^2[k_1^2(4\alpha_1^2-1)+4\alpha_1^2]+4\beta^3(\gamma k_1)[k_1^4+2k_1^2(4\alpha_1^2-3)+16(1-2\alpha_1^2)]+\beta^4F(k_1)=0 \quad (5.1)$$

$$F(k_1)=k_1^6-8k_1^4+8k_1^2(3-2\alpha_1^2)-16(1-\alpha_1^2) \quad (5.2)$$

$F(k_1)$ becomes $f(k_1)$ if the medium is undamped; that is $\gamma = 0$. By determining the leading coefficients of the Sturm's function for 5.1, the conditions that all its four roots are real combine to give:

$$k_1^4(5-12\alpha_1^2)+2k_1^2(50\alpha_1^2-9)+8(7+4\alpha_1^2)>0. \quad (5.3)$$

Interestingly, 5.3 is satisfied for all $k_1\ \varepsilon(0,1)$ and $3\alpha_1^2 = 1$. The inequality 5.3 thus, suggests the condition which ensures the attenuation of the Rayleigh surface waves passing through a damped elastic medium.

The relationship between δc and γ had been derived[3] using Taylor's expansion method. By neglecting terms of order γ^2, this relationship is of the form $\delta c = 0.22\gamma$. We have improved on this approach by neglecting terms of order γ^3. Also taking $3\alpha_1^2 = 1$, the improved relationship becomes:

$$\gamma=\left(\frac{\beta}{14}\right)\left[\frac{3k_1^6-23.5k_1^4-56k_1^2+32}{3k_1^5+10k_1^3+16k_1}\right] \quad (5.4)$$

$$k_1=c_0/\beta+\frac{i\delta c}{\beta}=k_0+\frac{i\delta c}{\beta} \quad (5.5)$$

k_o is the real factor of 5.2. With $\beta = 2.75$km/sec, 5.4 and 5.5

give $\gamma = 3.9H10^{-3}$km/sec and $k_o = 0.9$. These values are in agreement with established values[1,2,7,8].

In general therefore, with $\beta = 2.75$km/sec, the following results were computed from 5.4 and 5.5: (i) $\delta c = 0.15\gamma$ (ii) $\delta c = 0.21\gamma$ (iii) $\delta c = 0.316\gamma$, (i) and (ii) correspond to $k_1 > 1$ and (iii) corresponds to $k_1 < 1$. Because, the speed c_o of the surface wave satisfies the inequation $c_o < \beta < \alpha$, only the data (iii) gives the realistic representation of the surface waves. In the case of (i) and (ii), the activities of the corresponding seismic waves are no longer confined to the interface. Instead, they induce reflection processes among the corresponding body waves[4,9].

6. Conclusions:

We have shown that a generalized Rayleigh's function can be used to analyse the possible propagation velocities and attenuation of the small amplitude stress waves in elastic solids. This is interesting for the disturbances in solids are usually damped to varying degree.

Further, within the limits of the realistic analytic approximations, the application of this model is confined to the range of frequencies associated with single frequency microseisms. It is noted that there are numerous studies on the phenomena of surface waves appearing in recent publications. However, researches relating to the microseismic frequencies appear to be presently scanty. Consequently, few recently acquired data exist which appear to represent improvements on the existing results. Within this limitation, the model is tested with existing seismic records. Interestingly, equation 5.4 and 5.5 model the attenuation of small amplitude stress waves to a reasonable degree of accuracy.

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