J. Nig. Ass. Math. Phys.
Vol. 2 (1998)

NEW GROUP SYMMETRIES OF THE BURGER'S EQUATION: $U_t + UU_x + U_{xx} = 0$.

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ABSTRACT

Writing the Burger's equation in a conserved form, we obtain a system of partial differential equations. It is shown that this system admits new symmetries in the sense of Bluman et al.

INTRODUCTION

The study of symmetry groups of differential equations of mathematical physics, using group theoretic methods has attracted a great deal of attention since the pioneering work of Sophus Lie Olver[1] and others[2] have developed methods for determining the Lie – Backlund symmetries of partial differential equations; in a systematic way.

Moreover, Bluman et al[3] have shown that certain partial differential equations admit new classes of symmetries. These new symmetries were shown to be neither point symmetries nor Lie – Backlund symmetries. In this paper, we consider the Burgers equation $U_t + UU_x + U_{xx} = 0$. This is an important partial differential equation in non – linear wave theory[4]. The Lie – Backlund type symmetries have been computed and it was shown[5] that the infinitesimal symmetry algebra is five dimensional.

Here, we transform the Burger's equation into a system of partial differential equations in conserved form and then we show that this system admits new group symmetries in the sense of Bluman et al.

In the next section we give a brief introduction of the jet – bundle formalism leading to the prolongation formula for determining symmetry groups. Section III introduces the concepts of new symmetries and in section IV we subject the Burgers equation to the new symmetry tests. Section V contains our results and conclusions.

II. THE PROLONGATION FORMULA

Consider a system Σ of partial differential equations in m independent variables and n dependent variables. Let $M = \mathbb{R}^m$ with coordinates $x = (x_1, x_2, \ldots, x_m)$ represent the space of independent variables and $N = \mathbb{R}^n$ with coordinates $u = (u^1, u^2, \ldots, u^n)$ represent the space of dependent variables.

The solutions U = f(x) of Σ will be identified with their graphs (x,f(x)) in M x N.

A local group of transformations G acting on M X N such that solutions of Σ are transformed by G to other solutions of Σ will be called a symmetry groups of Σ.

Now, Let f: M $\rightarrow$ N be a function of m independent variables. We form a space containing all orders of partial derivatives of f. There are ${}^{m+r-1}C_r$ different r^{th} order partial derivatives of f. For each $J = (j_1 \ldots j_m)$, $j_i \geq = 0$ and $|J| = j_1 + \ldots + j_r$ we write

$$\partial_J = \frac{\partial^{|J|}}{\partial x_i^{j} \ldots\ldots \partial x_m^{j}} \qquad (1)$$

Let $U_r = \mathbb{R}^{n_r}$ with coordinates $U^I_J = \partial_J f^I(x)$, I = 1, ...,n where n_r is the number of the components U^I_J.

Finally, we put $U^{(r)} = U \times U_1 \times \ldots \times U_r$ to be the space of all partial derivatives of f of order $\leq$ r. Thus, corresponding to any function f :N $\rightarrow$N, is another function $pr^{(r)} f : M \rightarrow U^{(r)}$.

$pr^{(r)} f$ is called the r^{th} prolongation of f and it is defined by the equations

$$U^I_J = \partial_J f^I(x) \qquad (2)$$

M x $U^{(r)}$ is the r – jet space.

Suppose now we denote the partial differential equations under study by

$$\Delta(x,u^{(r)}) = 0,\ u^{(r)} \in U^{(r)} \qquad (3)$$

A solution of the equation is a smooth function f: N$\rightarrow$N such that $\Delta(x, pr^{(r)} f(x)) = 0$. Now, let G be a group of transformations and α be an infinitesimal generator of a one – parameter subgroup of G. We define the r^{th} – prolongation of α as the infinitesimal generator of the prolonged one – paramater subgroup $pr^{(r)} \exp(t\alpha)$ i.e

$$pr^{(r)}\alpha = \frac{d}{dt}\Big|_{t=0} pr^{(r)}\{\exp(t\alpha)\} \qquad (4)$$

The infinitesimal criterion for G to be a symmetry group of the equation Δ = 0 is the following:

" If for every infinitesimal generator α of G, $pr^{(r)}\alpha\, \Delta(x,u^{(r)}) = 0$ whenever $\Delta(x, u^{(r)}) = 0$, then G is a symmetry group of the equation Δ = 0".

Thus, once we have a formula for finding the prolongation of a vector field, we can determine the symmetry groups of a given system of partial differential equations. One such formula is the following, Olver[5].

Let α be a smooth vector field on M x N given by

$$\alpha = \sum_{i=1}^{m} \zeta^i (x.u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{n} \phi_1 (x,u) \frac{\partial}{\partial u_1} \quad (5)$$

Then, the r^{th} prolongation of α is the vector field given by:

$$pr^{(r)}\alpha = \alpha + \sum_{1=1}^{n} \sum_{j} \Phi_l^j (x, u^{(r)}) \frac{\partial}{\partial u_j^l} \quad on\ M \times U^{(r)} \quad (6)$$

Where the sum $\sum_j$ runs over all J with $|J| = j_1 + j_2 + \ldots + j_m \leq r$.

The functions Φ_1^j are given by

$$\Phi_1^j = D^j (\Phi^1 - \sum u_i^1 \zeta_i) + \sum u_{j,i}^1 \zeta_i \quad (7)$$

$$where \quad u_i^1 \frac{\partial u^1}{\partial x_i} \quad and \quad u_{j,i}^1 = \frac{\partial u_j^1}{\partial x_i}, \quad u_j^1 = \partial_j u^1 \quad (8)$$

$$D^J = D_1^{j1} o D_2^{j2} o \ldots o D_m^{jm} \quad (9)$$

$$and \quad D_i = \frac{\partial}{\partial x_i} + \sum \sum u_{j,i}^1 \frac{\partial}{\partial u_j^1} \quad (10)$$

D_i is the total derivative operator.

III CONCEPT OF NEW SYMMETRIES

Given a partial different equation S of order m in $n \geq 2$ independent variables, one re-writes it in conserved form. One then introduces a potential ϕ (for example) to obtain a system T of pdes, of two dependent variables viz:- ϕ and the original variable u (say). If T admits a one – parameter Lie group of point transformations, this group maps a solution of T into another solution of T and so induces a mapping of a solution of S into another solution of S. Therefore, the group so formed is a symmetry group of the pde S.

A new symmetry group of S arises if and only if any of the infinitesimal of system T depends especially on ϕ. See Ref [3] for details.

IV THE BURGER'S EQUATION

The Burger's equation is:

$$U_t + UU_x + U_{xx} = 0 \quad (11)$$

We re – write this in the conserved form as:

$$\partial F/\partial x - \partial G/\partial t = 0 \quad (12)$$

where $F = {}^1/_2 U^2 + U_x$ and $G = -U$

Thus, the associated system T is:

$$\partial F/\partial t = {}^1/_2 U^2 + U_x \quad (13)$$

$$\partial\phi/\partial x = - U \quad (14)$$

where $\phi = \phi(x,t,u)$ is some potential. T is now a system of pdes with two dependent variables ϕ and U and two independent variables x and t. We apply the prolongation formula to compute the vector fields which take the general form (on $X \times U \approx \mathbb{R}^2 \times \mathbb{R}^2$):

$$\alpha = \zeta_1\partial/\partial x + \zeta_2\partial/\partial t + \tau_1\partial/\partial u + \tau_2\,\phi\partial/\partial u\phi \quad (15)$$

where $\zeta_i = \zeta_i\,(x,t,u,\phi)$, $\tau_i = \tau_i\,(x,t,u,\phi)$, $i = 1,2$ (16)
in general

The solution set is a subvariety of $X \times U^{(2)}$

i.e $\Delta u = \phi_t - u^2/2 - u_x$ (17)

$\Delta\,\phi = \phi_x + u$ (18)

We shall obtain $Pr^{(2)}\ \alpha\Delta$

Recall that in this case we have from equation (6)

$$Pr^{(r)}\ \alpha = \alpha + \sum_J \Phi^J\,(x,u^{(r)})\,\partial/\partial u_J + \sum_J \Phi^J\,(x, u^{(r)})\,\partial/\partial\phi_J$$ so that

$$Pr^{(2)}\alpha = \alpha + \tau^x{}_1\,\partial/\partial u_x + \tau_1{}^t\,\partial/\partial u_t + \tau_2{}^x\partial/\partial\phi_x + \tau_2{}^t\partial/\partial\phi_t \quad (19)$$

since the other derivatives do not appear in our system of pdes.

Now, using the infinitesimal criterion $Pr^{(2)}\ \alpha\Delta = 0$ we have:

$$\tau_1 u u_x - \tau_1{}^x + \tau_2{}^t = 0 \quad (20)$$

$$\tau_1 + \tau_2{}^x = 0 \quad (21)$$

Whenever $\Delta = 0$

$$\tau_1{}^x = D_x\,(\tau_1 - u_x\zeta_1 - u_t\zeta_2) + u_{xx}\zeta_1 + u_{tx}\zeta_2$$

$$D_x = \partial/\partial x + u_x\,\partial/\partial u + u_{xx}\partial/\partial u_x + u_{xt}\partial/\partial u_t$$

$$\tau^t{}_2 = D_t\,(\tau_2 - \phi_x\zeta_1 - \phi_t\zeta_2) + \phi_{xt}\zeta_1 - \phi_{tt}\zeta_2$$

$$D_t = \partial/\partial t + \phi_t\,\partial/\partial\phi + \phi_{tt}\partial/\partial\phi_t + \phi_{tx}\partial/\partial\phi_x$$

$$\tau^x{}_2 = D_x\,(\tau_2 - \phi_x\zeta_1 - \phi_x\zeta_2) + \phi_{xx}\zeta_1 + \phi_{tx}\zeta_2$$

$$D_x = \partial/\partial x + \phi_x\,\partial/\partial\phi + \phi_{xx}\partial/\partial\phi_x + \phi_{xt}\partial/\partial\phi_t$$

$$\therefore \quad \tau^x{}_1 = D_x\tau_1 - u_x D_x\zeta_1 - u_t D_x\zeta_2$$

$$= \tau_{1x} + u_x\tau_{1u} - u_x\zeta_{1x} - u^2{}_x\zeta_{iu} - u_t\zeta_{2x} - u_t u_x\zeta_{2u}$$

$$\tau^x{}_2 = D_t\tau_1 - \phi_x D_t\zeta_1 - \phi_t D_t\zeta_2$$

$$= \tau_{2t} + \phi_t\tau_{2\phi} - \phi_x\zeta_{1t} - \phi_x\phi_t\zeta_{1\phi} - \phi_t\zeta_{2t} - \phi_t{}^2\zeta_{2\phi}$$

and

$$\tau^t{}_2 = D_x\tau_2 - \phi_x D_x\zeta_1 - \phi_t D_x\zeta_2$$

$$= \tau_{2x} + \phi_x\tau_{2\phi} - \phi_x\zeta_{1x} - \phi_x{}^2\zeta_1 - \phi_t\zeta_{2x} - \phi_t\phi_x\zeta_{2\phi}$$

Putting these back into (20) and (21), we get:

$$\tau_1 u u_x - \tau_{1x} - u_x\tau_{1u} + u_x u\zeta_{1x} + u_x{}^2\zeta_{1u} + u_t\zeta_{2u} + u_t u_x\zeta_{2u}$$
$$+ \tau_{2t} + \phi_t\tau_{2\phi} - \phi_x\zeta_{1t} - \phi_x\phi_t\zeta_{1\phi} - \phi_t\zeta_{2t} - \phi_t{}^2\zeta_{2\phi} = 0$$

And

$$\tau_1 - \tau_{2x} + \phi_x\tau_{2_\phi} - \phi_x\zeta_{1x} - \phi^2{}_x\zeta_{1_\phi} - \phi_t\zeta_{2x} - \phi_t\phi_x\zeta_{2_\phi} = 0$$

Next replace u_t by $-uu_x - u_{xx}$ and ϕ_t by $u^2/2 + u_x$ noting that

$$u_x = -\phi_{xx}\ , \quad \phi_x = -u$$

Now equating the coefficients of the various partial derivatives of u, yields the following set of symmetry equations, after some eliminations:

$$\tau_1 u - \tau_{iu} + \zeta_{1x} + \tau_{2_\phi} + u\zeta_{1_\phi} + \zeta_{2t} = 0 \qquad (22)$$

$$-\tau_{1x} + \tau_{2t} + (u^2/2)\,\tau_{2_\phi} + u\zeta_{1t} + (u^3/u)\ \zeta_{1_\phi} - (u^2/2)\,\zeta_{2t} = 0 \qquad (23)$$

$$\tau_1 - \tau_{2x} - u\tau_{2_\phi} + u\zeta_{1x} + u^2\zeta_{1_\phi} = 0 \qquad (24)$$

$\zeta_{2_\phi} = \zeta_{2u} = \zeta_{2x} = \zeta_{1u} = 0$. This implies $\zeta_2 = \zeta_2(t)$.

V. RESULT AND CONCLUSION

We solved the above system of non – linear partial differential equations (22) – (24) seeking an explicit ϕ - dependence.

After some manipulations one gets, for example,

$$\tau_2 = \mathbf{F}(t)\phi - u\,\mathbf{G} - \mathbf{H} \qquad (25)$$

where:

$\mathbf{F}(t) = \zeta_2(t)$, $\mathbf{G} = \mathbf{G}(x,t,u,\phi)$ and $\mathbf{H} = \mathbf{H}(x,t,u,\phi)$.

The remaining infinitesimal parameters can be obtained similarly. Clearly, the Burger's equation admits a class of new symmetries as defined in section III above.

It must be noted that the choice of the conserved form is not necessarily unique. Hence, it is quite possible to obtain different new symmetries for this equation considered in this paper.

The computation of the new infinitesimal generators of the lie algebra and possible symmetry solutions are in progress and will be reported elsewhere

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