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PERTURBATIVE QUANTUM CHROMODYNAMICS

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ABSTRACT
Introductory covariant perturbation theory for quantum chromodynamics is discussed, with examples to serve as qualitative illustrative applications.

1. INTRODUCTION

Quantum chromodynamics (QCD) is the theoretical model of strong interactions in which quarks, the constituents of hadrons, are assumed to be confined inside the hadrons. The two types of hadrons are baryons and mesons. QCD is based on the standard $SU(3)_C \times SU(2)_L \times U(1)$ model, and has its experimental foundation in the results of deep inelastic scattering of leptons on hadrons, in which it is found that at high momentum transfer, that is, at distances smaller than the hadronic size, hadrons behave as if they consist of point-like weakly interacting particles (quarks and gluons). This observation is essentially the notion of asymptotic freedom. However, at larger distances, that is, at distances of hadronic size, the interactions between these constituent particles become so strong that the particles cannot escape from the hadron. This is the notion of quark and gluon confinement. Thus, hadrons are viewed as bound states of quarks whose interactions are mediated by gluons, the gauge field of the colour SU(3) group.

Let us note that according to the standard model, the universe comprises the spin $\frac{1}{2}$ fermions: leptons and quarks, each set being six in number. The leptons are the electron, ν_e, muon, ν_μ, tau, and ν_τ; while the types, that is, flavours of quarks are down (d), up (u), strange (s), charm (c), bottom or beauty (b), and top (t). Only the last type of quark has not been discovered. These fermions can be arranged in three "generations" or "families". The first generation are the leptons, e^- and ν_e; and the quarks, d and u. The second generation are the leptons, μ^- and ν_μ; and the quarks, s and c. The third generation are the leptons, τ^- and ν_τ; and the quarks, b and t. The mesons are made up of one quark and one antiquark. In the meson octet, we have the examples $\pi^+(u\bar{d})$ and $\pi^-(d\bar{u})$. Each baryon is made up of three quarks, for example, p(uud) and n(udd) in the baryon octet. To circumvent problems associated with the Pauli principle, for example, in p(uud), each quark can exist in any of the colours red, blue, and green. However, only combinations of colours which are additively colourless constitute physical particles like hadrons. The perturbative approach is one of the two main methods employed in quantum chromodynamics. In a nonperturbative method of QCD, the finite-size discrete lattice approximates continuous four-dimensional space-time. This is lattice quantum chromodynamics.

2. COVARIANT PERTURBATIVE QCD

To obtain the QCD Lagrangian, let us begin by introducing spinor fields forming an SU(3) triplet:

$$\psi^a(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \\ \psi^3(x) \end{pmatrix}$$

The free Lagrangian for this triplet has the form

$$L(\psi^a(x),\ \partial_\mu\psi^a(x)) = i\tilde{\psi}^a\gamma^\mu\partial_\mu\psi^a - M\tilde{\psi}^a\psi^a \qquad (1)$$

Here, the (covariant) vector $\partial_\mu = (\partial_0, \vec{\nabla})$; and γ_μ are the Pauli-Dirac matrices

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$k = 1, 2, 3$, with σ_k being the 2 x 2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \text{and}\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The conjugate spinor is $\tilde{\psi}^a = \psi^{a+}\gamma^0$, and M is the mass of the particle described by the spinor field. For later use, we define the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The Lagrangian in eq (1) is invariant under the global non-Abelian colour SU(3) group of transformations:

$$\psi'^a(x) = [\exp(-\tfrac{1}{2}ig\lambda_m\epsilon_m)]_{ab}\psi^b(x) \qquad (2a)$$

$$\tilde{\psi}'^a(x) = \tilde{\psi}^b(x)[\exp(\tfrac{1}{2}ig\lambda_m\epsilon_m)]_{ba} \qquad (2b)$$

In these equations, g is a coupling constant, and the Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (3a)$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};\ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix};\ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (3b)$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix};\ \lambda_8 = 3^{-\frac{1}{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad (3c)$$

with $\lambda_k/2i$ being the generator of the fundamental representation of SU(3). The λ_k's obey

$$[\lambda_k, \lambda_j] = 2if_{kjn}\lambda_n; \ \mathrm{Tr}[\lambda_k \lambda_j] = 2\delta_{kj} \qquad (3d)$$

with f_{kjn} being the totally antisymmetric structure constants and ε_m the constant parameters of SU(3). It follows from equations (2a) and (2b) that the infinitesimal transformations are given by

$$\delta\psi^a(x) = -\tfrac{1}{2}ig(\lambda_m)^{ab}\varepsilon_m\psi^b(x)$$

$$\delta\tilde{\psi}^a(x) = \tfrac{1}{2}ig(\lambda_m)^{ba}\varepsilon_m\tilde{\psi}^b(x)$$

As usual, summation is to be carried out on repeated indices. Therefore, the generators of the group are

$$T^k_{ab} = -\tfrac{1}{2}ig(\lambda_k)_{ab}.$$

With the aid of eq (3d), one finds that

$$[T^k, T^l]^{ab} = -\tfrac{1}{4}g^2[\lambda_k, \lambda_l]^{ab}$$

$$= -\tfrac{1}{4}g^2(2i)f_{klm}(\lambda_m)^{ab} = gf_{klm}T^m_{ab}$$

We first note that a globally invariant Lagrangian can be made locally (gauge) invariant, that is, invariant under the group of transformations

$$\delta u_i(x) = T^k_{ij}\,\varepsilon_k(x)u_j(x)$$

by introducing the gauge fields $A^k_\mu(x)$, which are then found to enter L in the combination

$$\nabla_\mu u_i = \partial_\mu u_i - T^k_{ij}u_jA^k_\mu$$

known as the covariant derivative. With u_j replaced by the spinor fields ψ^a, the covariant derivative is

$$\nabla_\mu\psi^a(x) = \partial_\mu\psi^a(x) + \tfrac{1}{2}ig(\lambda_m)^{ab}\psi^b(x)V^m_\mu(x),$$

the gauge fields $V^m_\mu(x)$ forming an octet called gluon fields. We may note that the simplest locally invariant Lagrangian for gauge fields is a function of $F^k_{\mu\nu}$ only. It is the Yang-Mills Lagrangian

$$L_{YM} = -\tfrac{1}{4}F^k_{\mu\nu}F^k_{\mu\nu}$$

with

$$F^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu - \tfrac{1}{2}gf_{npk}(V^n_\mu V^p_\nu - V^n_\nu V^p_\mu)$$

being the gluon field tensor. The infinitesimal transformations of gauge fields are given by

$$\delta V^m_\mu(x) = gf_{npm}V^p_\mu(x)\varepsilon_n(x) + \partial_\mu\varepsilon_m(x).$$

The total locally invariant Lagrangian, which is the Lagrangian for quantum chromodynamics, is then

$$L = i\tilde{\psi}^a\gamma_\mu\partial_\mu\psi^a - M\tilde{\psi}^a\psi^a - \tfrac{1}{2}g\tilde{\psi}^a\gamma_\mu(\lambda_m)^{ab}V^m_\mu\psi^b$$
$$-\tfrac{1}{4}[\partial_\mu V^m_\nu - \partial_\nu V^m_\mu - \tfrac{1}{2}gf_{npm}(V^n_\mu V^p_\nu - V^n_\nu V^p_\mu)]^2 \qquad (4)$$

Given the Lagrangian density L, the quantity

$$J^k_\mu(x) = \partial L / \partial A^k_\mu(x)$$

is called the current. We have current conservation if and only if

$$\partial_\mu J^k_\mu(x) = 0,$$

k = 1,2,...,n. Corresponding to the Lagrangian in eq (4) is the conserved current

$$J^m_\mu(x) = (\partial L(x)/\partial \nabla_\mu \psi^a)(\tfrac{1}{2}ig(\lambda^m)^{ab}\psi^b(x))$$
$$- 2g(\partial L(x)/\partial F^n_{\mu\nu})f^{mpn}V^p_\nu(x) \qquad (5)$$

The Greek index $\mu = 0,1,2,3$ is the Lorentz group index, while the index m enumerates the generators of the colour SU(3) group.

3. COVARIANT PERTURBATION THEORY

Since the effective (or running) coupling constant $\alpha_s(q^2)$ is small for large momentum transfer, perturbation theory is applicable for large q^2 transfer. However, for large distances, perturbation theory is no longer applicable since $\alpha_s(q^2)$ becomes large. The transition amplitude for the gauge field can be written as

$$S = \int \prod_{x,k} DA^k_\mu(x) \det M_a \exp\{i\int dx\,[-\tfrac{1}{4}F^k_{\mu\nu}F^k_{\mu\nu}$$
$$- \tfrac{1}{2}(\partial_\mu A^k_\mu)^2/\alpha \qquad (6)$$

where

$$M_a = \Box\delta_{kl} + g\,\varepsilon_{klm}(A^m_\mu\partial_\mu + \partial_\mu A^m_\mu).$$

The amplitude in eq (6) is said to be written in the α-gauge. In the path integral in eq (6),

$$\prod_{x,k} DA^k_\mu(x) \equiv \lim_{N\to\infty} (\prod_k \prod_{i=1}^{N} dA^k_\mu(x_i))$$

is the integration measure. In eq (4), the part of the Lagrangian involving the gluon field can be written in the α-gauge as the sum of two Lagrangians

$$L_o = -\tfrac{1}{4}(\partial_\mu V^m_\nu - \partial_\nu V^m_\mu)^2 - \tfrac{1}{2}(\partial_\mu V^m_\mu)^2/\alpha$$

and

$$L_{os} = gf_{npm}(\partial_\mu V^m_\nu)V^n_\mu V^p_\nu - \tfrac{1}{4}g^2 f_{jpm} f_{lnm} V^j_\mu V^p_\nu V^l_\mu V^n_\nu$$

where the former represents a noninteracting gluon field, while the latter represents the self-interaction of the gluon fields. In the path-integral formulation of quantum field theory, a Green's function is written in terms of the vacuum expectation value of the chronological product of field operators, where the term "chronological" refers to the ordering of the time argument of the field operators such that it increases from right to left. For example, with T denoting the chronological product, the propagator of two operators of a scalar field, that is, the two-point Green's function, is $i\langle 0|\,T(\phi(x)\phi(y)\,|0\rangle$. A most useful way of deriving Green's functions is to introduce an auxilliary current $J_i(x)$ for each field $u_i(x)$, to obtain the generating functional W(J). For n fields, this is defined as

$$W(J) = \int\prod_x D\mu(u_1(x),\ldots,u_n(x))\exp\{i\int dx[L(x) + u_1(x)J_1(x) + \ldots + u_n(x)J_n(x)]\}$$

The identity

$$\exp(\sum_k a_k u^k + Ju) = \exp[\sum_k a_k(\delta/\delta J)^k]\exp(Ju)$$

where $\delta/\delta J$ is the functional derivative with respect to the current, gives the quantum chromodynamics generating functional in the α-gauge as

$$W = R\int DV^m_\mu(x)D\tilde{\psi}^a(x)D\psi^a(x)D\tilde{c}^m(x)Dc^m(x)\exp\{i\int dx[A + B + C] \tag{7}$$

where

$$R = \exp\{i\int dxL_I(i^{-1}\delta/\delta J^m_\mu(x)\cdots)\}$$

$$A = \tfrac{1}{2}V^m_\mu\delta^{mn}(\Box g_{\mu\nu} - (1-\alpha^{-1})\partial_\mu\partial_\nu)V^n_\nu(x)$$

$$B = \tilde{\psi}^a(x)\delta^{ab}(i\gamma_\mu\partial_\mu - M)\psi^b(x) + \tilde{c}^m(x)\delta_{mn}\Box c^n(x)$$

and

$$C = J^m_\mu(x)V^m_\mu(x) + \tilde{\eta}^a(x)\psi^a(x) + \tilde{\psi}^a(x)\eta^a(x) + \tilde{\chi}^m(x)c^m(x) + \tilde{c}^m(x)\chi^m(x)$$

By performing certain transformations, the generating functional can also be written in the form:

$$\begin{aligned} W(J^m_\mu, \tilde{\eta}^a, \eta^a, \tilde{\chi}^m, \chi^m) &= R\exp\{-i\int dxdy[\tfrac{1}{2}J^m_\mu(x)(K^{mn}_{\mu\nu}(x-y))^{-1}J^n_\nu(x) \\ &\quad + \tilde{\eta}^a(x)P^{-1}_{ab}(x-y)\eta^b(y) \\ &\quad + \tilde{\chi}^m(x)P^{-1}_{mn}(x-y)\chi^n(y) \end{aligned} \tag{8}$$

where $K^{mn}_{\mu\nu}(x-y)$ and $P^{-1}_{ab}(x-y)$ are differential operators determined by the form of the free Lagrangian density. Since the generating functional can also be written as

$$W(J) = \exp[i\int dxL_I(i^{-1}\delta/\delta J_1(x),\ldots,i^{-1}\delta/\delta J_n(x)]\exp(G),$$

where

$$G = -\tfrac{1}{2}i\int dxdyJ_i(x)K^{-1}_{ij}(x-y)J_j(y),$$

then, within the framework of perturbation theory, the S matrix elements are given by

$$S(u^0_k) = \exp(G_1)\exp(G_2)\Big|_{J_1=\ldots=J_n=0}$$

where

$$G_1 = i\int dxL_I(i^{-1}\delta/\delta J_1(x),\ldots,i^{-1}\delta/\delta J_n(x),$$

$$G_2 = -i\int dxu^0_k(x)J_k(x) - \tfrac{1}{2}i\int dxdyJ_i(x)K^{-1}_{ij}(x-y)J_j(y),$$

and $u^0_k(x)$ are arbitrary functions. The S matrix elements are given in terms of eq (8) by

$$S(V^m_{\mu 0},\ldots) = R\exp(A_1 + A_2)\Big|_{J^m_\mu=\ldots=0} \tag{9}$$

where

$$A_1 = -i \int dx (V^m_{\mu 0} J^m_\mu + \tilde{\psi}^a_0 \eta^a + \tilde{\eta}^a \psi^a_0$$

$$A_2 = -i \int dxdy \Big[\tfrac{1}{2} J^m_\mu (x) (K^{mn}_{\mu\nu}(x - y))^{-1} J^n_\nu (y)$$

$$+ \tilde{\eta}^a(x) P^{-1}_{ab}(x - y)\, \eta^b(y)$$

$$+ \tilde{\chi}^m(x) P^{-1}_{mn}(x - y)\, \chi^n(y)$$

$$[K^{kl}_{\mu\nu}(x - y)]^{-1} = -\delta_{kl}(2\pi)^{-4} \int k^{-2} dk \big[g_{\mu\nu}$$

$$- (1 - \alpha)(k_\mu k_\nu / k^2) \big] e^{-ik(x - y)}$$

$$P^{-1}_{ab}(x - y) = \delta_{ab}(2\pi)^{-4} \int dp\, e^{-ip(x - y)} (\not{p} - M)$$

$$P^{-1}_{kl}(x - y) = -\delta_{kl}(2\pi)^{-4} \int dk\, e^{-ik(x - y)} k^{-2}$$

$$K^{-1}_{ab}(x - y) = \delta_{ab}(2\pi)^{-4} \int dk\, e^{-ik(x - y)} (k^2 - M^2)$$

and

$$\not{p} = p_\mu \gamma_\mu = i \partial_\mu \gamma_\mu = i \not{\partial}$$

When the differential operators appearing in eq (9) are expanded in series in the coupling constant g, we obtain the following expression within the perturbation theoretic framework of QCD:

$$S(V^a_{\mu 0}, \ldots) = \Big[1 + \tfrac{1}{2} g \int dx B_1 - g f_{npm} \int dx B_2$$

$$- \tfrac{1}{4} g^2 i \int dx f_{jpm} f_{lnm} B_3 + g \int dx f_{pnm} B_4$$

$$+ \ldots \Big] \exp(B_5) \Big|_{J^m_\mu = \ldots = 0} \qquad (10)$$

where

$$B_1 = (\delta / \delta \eta^a(x))\, \gamma_\mu (\lambda_m)^{ab} (\delta / \delta \tilde{\eta}^b(x)) (\delta / \delta J^m_\mu (x))$$

$$B_2 = \partial_\mu (\delta / \delta J^n_\nu (x)) (\delta / \delta J^n_\mu (x)) (\delta / \delta J^p_\nu (x))$$

$$B_3 = (\delta / \delta J^j_\mu (x)) (\delta / \delta J^p_\nu (x)) (\delta / \delta J^l_\mu (x)) (\delta / \delta J^n_\nu (x))$$

$$B_4 = \partial_\mu (\delta / \delta \chi^m(x)) (\delta / \delta J^n_\mu (x)) (\delta / \delta \tilde{\chi}^p(x))$$

and

$$B_5 = -i \int dx (V^m_{\mu 0}(x) J^m_\mu (x) + \tilde{\psi}^a_0 \eta^a + \tilde{\eta}^a \psi^a_0)$$

$$-i \int dxdy \Big[\tfrac{1}{2} J^m_\mu (K^{mn}_{\mu 0})^{-1} J^n_\nu + \tilde{\eta}^a P^{-1}_{ab} \eta^b + \tilde{\chi}^m P^{-1}_{mn} \chi^n.$$

We may note in eq (10) that apart from the fact that functions of free particles (that is, quarks and gluons) occur in various combinations, propagators (for quarks, gluons, ghost fields), and interaction vertices (for example, for three gluons, one gluon and two ghosts) are evident. The expressions for the propagators are listed in table 1, which is taken from ref [1]. Correspondence rules for the amplitudes in momentum representation have been derived, and are summarized in table 2 (see ref [1]). Note that the photon, the gauge field of quantum electrodynamics (QED), is replaced in quantum chromodynamics by eight gluons which, in contradistinction from photons, are also self-interacting. The other differences with QCD are as follows: The non-Abelian nature of QCD manifests itself by the presence of structure constants and generators of the gauge group in the expressions for QCD. Moreover, the fact that gluons are self-interacting, results in additional interaction vertices such as three-gluon and four-gluon vertices. On account of this, we find, in QCD, diagrams with no analogues in QED.

Table 1: Feynmann diagrams for the propagators

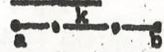
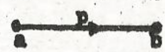
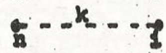
Propagator of	Analytic expression	Diagram
scalar field	$-\delta_{ab}/(k^2+m^2)$	
spinor field	$-\delta_{ab}/(p-M)$	
ghost field	δ_{nl}/k^2	
gauge field	$k^{-2}\delta_{nl}(g_{\mu\nu} - (1-\alpha)k_\mu k_\nu/k^2)$	

Table 2: Correspondence rules for amplitudes

Physical state	Mathematical expression	Diagram
1. Quark in the initial state	$v_r^{(-)}(p)$	
2. Anti-quark in the initial state	$\tilde{v}_r^{(-)}(p)$	
3. Quark in the final state	$\tilde{v}_{r'}^{(+)}(p')$	
4. Anti-quark in the final state	$v_{r'}^{(+)}(p')$	
5. Gluon in the initial or final state	ε_μ^k	
6. Motion of virtual quark from a to b	$-\delta_{ab}/(\not{p}-m)$	
7. Motion of virtual anti-quark from a to b	$\delta_{ab}/(\not{p}+m)$	
8. Motion of virtual gluon between the states αa and βb	$k^{-2}\delta_{ab}[g_{\alpha\beta} - (1-\alpha)k_\alpha k_\beta]$	
9. Motion of virtual ghost	δ_{ab}/k^2	
10. Quark-gluon interaction vertex	$g\gamma_\alpha(\lambda_a)^{cb}$	
11. Ghost-gluon interaction vertex	$igf_{abc}q_\alpha$	
12. Three-gluon interaction vertex	$-igf_{abc}[(r-q)_\alpha g_{\beta\gamma} + (p-r)_\beta g_{\alpha\gamma} + (q-p)_\gamma g_{\alpha\beta}]$	

We may note that renormalization refers to the removal of divergences from various quantities appearing in the theory. This process presupposes the existence of an intermediate regularization invariant under gauge transformations. Theories exist, however, with no invariant regularization, and they include those such that the gauge transformations of the fermion fields contain the matrix γ_5. Such theories have certain characteristics. As an example, the three-point vertex Green's function $\Gamma_{\mu\nu\alpha}$ can be directly calculated in a model invariant under the simplest group U_1 to give

$$i(p + q)_\alpha \Gamma_{\mu\nu\alpha}(p, q) = -(g^3/6\pi^2)\varepsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta$$

When the right-hand side of this equation is nonzero, such theories are said to display anomalies, meaning that different gauges are inequivalent. In this model, a unitary renormalized S-matrix does not exist. QCD does not display anomalies since its Lagrangian and currents do not contain the γ_5 matrix. In conclusion, we now mention a few processes to which perturbation theory can be applied.

4. SOME BASIC PROCESSES

Hard processes are processes in which either a large momentum transfer q^2 is involved in the final state of a particle (hadron, photon, lepton) or a system with large virtual or actual mass results in the final state. Perturbation theory is applicable to hard processes, of which the following are examples:

1. Deep inelastic lepton scattering on nucleons.
2. Lepton pair annihilation, for example, $e^+e^- \longrightarrow h$ and $e^+e^- \longrightarrow h + X$ where h stands for a hadron while X stands for all other particles.
3. Following the collision of hadrons, the production of
   (a) a muon pair with large effective mass, $pp \longrightarrow \mu^+\mu^- X$ (this is the Drell-Yan process);
   (b) charmed particle pairs, $pp \longrightarrow c\bar{c}X$;
   (c) π mesons with large momentum transfer, $pp \longrightarrow \pi X$.
4. The formation of jets. Particle fluxes which are concentrated within small solid angles are called jets, and these may be initiated by quarks and gluons.

In order to study a process in QCD, one first calculates the cross-sections for the subprocesses involving quarks and gluons, before going over to hadrons.

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