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ON QUANTUM CHAOS

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ABSTRACT
Given a closed Hamiltonian system of n degrees of freedom, with Hamiltonian $H(p,q) = T(p) + V(q) = E$, where E is a constant and $n \geqslant 2$, we now know that the system exhibits deterministic (classical) chaos, if H is nonlinear and completely nonintegrable. The question often arises: How should such a system be quantized, and is there any manifestation of the (classical) chaoticity in its quantal counterpart? Seeking answers by the method of reduction of numbers of degrees of freedom, we demonstrate that some attribute L (related to one of the momenta p) can be quantized, provided the coordinates q satisfy some relationships. These relationships are obtained by the method of the "Lax pair" - the potential V(q) must satisfy some "KdV equation" of "solitary waves". Moreover, the relationships reveal that the KdV engenders multiple Schroedinger operators and this is the manifestation of the chaoticity. Hence, such a quantum-mechanical system has multiple spectra and therefore it is noisy.

1. PRELIMINARY

Let the manifold M be the configuration space of a closed dynamical system with the generalized coordinates $q \equiv (q_1, q_2, \ldots, q_n) \in M$, $q_j = q_j(t)$, with $n \geqslant 2$ as the number of relevant degrees of freedom of the system. We assume that $dq_j/dt = \dot{q}_j$ exists for all j, and for all time t. To M corresponds a tangent bundle TM with 2n coordinates $(\dot{q},q)$. Suppose that the system has the Lagrangian L,

$$\left.\begin{array}{l} L: TM \longrightarrow \mathbb{R} \\ (\dot{q},q) \longrightarrow L(\dot{q},q) \end{array}\right\} \qquad (1)$$

Under the assumption that $p_j \equiv \partial L/\partial \dot{q}_j$ exists as the momentum conjugate to q_j, and through the Legendre transformation, we obtain the Hamiltonian function on the cotangent bundle T*M of M,

$$\left.\begin{array}{l} H: T^*M \longrightarrow \mathbb{R} \\ (p,q) \longrightarrow H(p,q) \end{array}\right\} \qquad (2)$$

We also take it that the phase space T*M is a bounded subset of $\mathbb{R}^{2n}$. The classical dynamics of the system is given by Hamilton's equations of motion (Goldstein, 1950):

$$\left.\begin{array}{l} dq_j/dt = \partial H/\partial p_j \\ dp_j/dt = -\partial H/\partial q_j \end{array}\right\} \qquad (3)$$

(A) Deterministic (classical) chaos: As a closed system, $H(p,q) = E$ is constant. If, for $n \gg 2$, H is nonlinear and $H = E$ is the only integral of motion, the system is completely nonintegrable (Akin-Ojo, 1990); the system exhibits deterministic (classical) chaos. That is, its orbit in T*M is sensitively dependent on initial conditions. The question that arises is: What is the behaviour of the quantal counterpart of the system, with the Hamiltonian operator $H_Q \equiv H(-i\partial/\partial q, q)$, or how does the classical chaos manifest itself in the quantal system? This paper is addressed to this question.

(B) Conjugacy of t and -H: As a preliminary, we show that just as p_j is the momentum conjugate to coordinate q_j, $-H$ is the '**momentum**' conjugate to 'coordinate' t. With $v_j \equiv dq_j/dt$, $p_j = \partial L/\partial v_j$, we have

$$v_j \partial L/\partial v_j - L = H(p,q) \tag{4}$$

(the repeated index denotes summation). Define $q_o \equiv t$, $v_o \equiv dt/d\tau$, gives $v_j = (dq_j/d\tau)(d\tau/dt) = q_j'/v_o$, where τ is some independent variable (such as proper time). Define (as in Dirac, 1964; Akin-Ojo, 1988):

$$L_o \equiv v_o L(v,q) = v_o L(q'/v_o, q) \tag{5}$$

with the all-important Action remaining invariant,

$$\begin{aligned} \text{Action} \equiv \int L_o d\tau &= \int v_o L(v,q) d\tau \\ &= \int L(v,q) v_o (d\tau/dt) dt \\ &= \int L(v,q) dt \end{aligned} \tag{6}$$

Then, the momentum p_o conjugate to $q_o \equiv t$, is given by

$$p_o = \partial L_o/\partial v_o = L + v_o \left[\partial L/\partial(q_j'/v_o)\right]\left[\partial(q_j'/v_o)/\partial v_o\right]$$

Noting that $\partial L/\partial(q_j'/v_o) = v_o(\partial L/\partial q_j') \equiv v_o \pi_j$, we obtain

$$\begin{aligned} p_o &= L + v_o^2 \pi_j(-q_j'/v_o^2) \\ &= L - \pi_j q_j' \\ &= L - (p_j/v_o)(v_j v_o) = L - p_j v_j \equiv -H \end{aligned} \tag{7}$$

(C) Reduction of degrees of freedom: Consequently, with $E \equiv -h$, if we solve

$$H(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n) + h = 0$$

to obtain

$$p_n + K(p_1, \ldots, p_{n-1}, q_1, \ldots, q_n, h) = 0 \tag{8}$$

this $K \equiv -p_n$ is a 'Kamiltonian' with q_n as 'time', so that Hamilton's equations (3) (now Kamilton's equations) are given by

$$dq_j/dq_n = \partial K/\partial p_j \tag{9a}$$

$u(x,t)$ satisfies some KdV equation such as

(i) $\quad u_t - u_x = 0,$

or

(ii) $\quad u_t + uu_x + u_{xxx} = 0.$

(There is an infinite **hierarchy** of KdV equations; solutions are required to obey $u \longrightarrow 0$ as $x \longrightarrow \infty$). For (ii), we have

$$u(x,t) = (3c)\mathrm{sech}^2(\tfrac{1}{2}c^{\frac{1}{2}}(x - t)) \tag{14}$$

where c is the speed of the solitary wave. And for (i), solution is trivial if x and t have infinite range (Lax, 1968). It is with (i) that we shall be concerned here: (i) is not trivial if x and t have finite ranges.

2. QUANTUM CHAOS

How does the classical chaoticity of H manifest itself in the corresponding quantum system? Consider a closed conservative system, with the nonlinear classical Hamiltonian function

$$H \equiv \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1,q_2) = E = \text{constant} \tag{15}$$

that is,

$$H(p,\pi,x,y) = \tfrac{1}{2}(p^2 + \pi^2) + V(x,y) = E$$

where H is defined on $(x,y) \in \mathbb{R}^2$. Solving for π, we have

$$\pi = \left\{2E - \left[p^2 + 2V(x,y)\right]\right\}^{\frac{1}{2}} = -K(p,x,y) \tag{16}$$

that is,

$$\pi = (2E - L)^{\frac{1}{2}} = -K(p,x,y)$$

where $L \equiv p^2 + u(x,y)$, and $u \equiv 2V$. Thus with $y \equiv \tau$, by eq (9), the classical dynamics of the system is given by

$$dp/d\tau = -\partial K/\partial x = -u_x/K$$
$$dx/d\tau = \partial K/\partial p = p/K \tag{17}$$

the solutions of which are expected to be chaotic.

Now, to quantize H is to quantize K, or K^2, which in turn is to quantize L, through the Schroedinger equation,

$$(-D^2 + u(x,\tau))\psi(x) = \lambda\psi(x) \tag{18}$$

where $D = d/dx$. This, as aforementioned, requires that $u(x,\tau)$ satisfy, nontrivially, the equation

$$u_x = u_\tau \tag{19}$$

in the finite configuration space M defined by $|x| < a < \infty$, and $|\tau| < b < \infty$. The solutions to eq (19) are in classes (i), (ii), and (iii) of possibilities:

(i) Eq (19), $u_x = u_\tau$, may lead to a tautology, $0 = 0$. This is the case in a linear system, such as $u(x,\tau) \equiv x^2 + \tau^2 + 2x\tau$, which we are not here concerned with.

(ii) Eq (19) gives $x = k\tau$ as the only solution in M. This is the case for nonlinear but integrable systems. For example,

$$u(x,\tau) = \tfrac{1}{2}(x^4 + \tau^4) + 3x^2\tau^2$$

with $u_x - u_\tau = 0$ gives

$$2x^3 + 6x\tau^2 = 2\tau^3 + 6x^2\tau$$

with the unique solution $(x - \tau)^3 = 0$, or $x = \tau$.

(iii) Eq (19) leads to multiple solutions

$$\tau^{(r)} = \nu_r(x), \tag{20}$$

$r = 1,2,3,\ldots$, in which case there are multiple operators

$$L^{(r)} \equiv D^2 + u(x, \nu_r(x)),$$

with multiple quantum spectra $\{\lambda_s^{(r)}\}$ given by

$$[-d^2/dx^2 + u(x, \nu_r(x))]\,\psi_s^{(r)}(x) = \lambda_s^{(r)}\psi_s^{(r)}(x) \tag{21}$$

This is the case for nonlinear nonintegrable systems. The multiplicity of spectra is the manifestation of the nonintegrability of H, that is, of deterministic, classical chaos. We take the Henon-Heiles potential (Henon, 1964) as an example:

$$u(x,\tau) = x^2 + \tau^2 + 2x^2\tau - \tfrac{2}{3}\tau^3.$$

The equation $u_x - u_\tau = 0$ gives

$$\tau^2 - (1 - 2x)\tau + (x - x^2) = 0$$

or

$$\tau^{(1),(2)} = \tfrac{1}{2}[(1 - 2x) \pm (1 - 8x + 8x^2)^{\frac{1}{2}}] \tag{22}$$

Here, there are two solutions provided that $8x^2 - 8x + 1 > 0$. This provision is feasible depending on the value of E in the equation

$$2H \equiv (p^2 + \pi^2) + 2V(x,y) = 2E.$$

3. REMARKS AND CONCLUSION

(i) In the Hamiltonian function $H(p,q) \equiv T(p) + V(q) = E =$ constant, where

$$T(p) \equiv \sum_{j=1}^{n} \tfrac{1}{2}p_j^2 \geqslant 0,$$

it is the case that the configuration space M defined by $V(q) \leqslant E$ is necessarily bounded for $|E| < \infty$. Thus on M, V(q) is defined; and outside of M, V(q) is zero as $|q| \longrightarrow \infty$, but it is not C^∞, so $\psi(x,\tau)$ is a pseudo wave. That is, our $u(x,\tau)$ is not a true solitary wave. We have only utilized (here, the **nontrivial**) equation $u_t - u_x = 0$, in the method of "Lax pair" used in nonlinear wave analysis, to exhibit the inherent multiplicity of Schroedinger equations for a nonlinear nonintegrable classical Hamiltonian on finite phase space T*M.

(ii) The recent analysis by Kaushal and others (Kaushal, 1991) shows that a Hamiltonian of the form

$$H \equiv \tfrac{1}{2}(p_1^2 + p_2^2) + V(x,y)$$

in which V(x,y) is anharmonic (and nonintegrable) in x and y **cannot** be quantized unless there is some inverse term such as 1/x in V(x, y). Again here, his analysis refers to cases in which x and y have infinite range. For finite range, work is in progress to show that this conclusion does not apply.

(iii) The hydrogenic atom in a very strong static magnetic field has received a lot of attention recently in the context of classical chaos. The transformed Hamiltonian, with $\mathcal{E} \propto E$, and angular momentum $l_z = 0$, is

$$H = \tfrac{1}{2}(p^2 + \pi^2) - \varepsilon(x^2 + \tau^2) + \tfrac{1}{8}x^2\tau^2(x^2 + \tau^2) = 2.$$

Here, $u_\tau - u_x = 0$ gives

$$\tau^2 = -x^2 \pm (x^4 + 8\varepsilon)^{\frac{1}{2}}$$

which, for $\varepsilon > 0$, has the multiple solutions

$$\tau = \pm[(x^4 + 8\varepsilon)^{\frac{1}{2}} - x^2]^{\frac{1}{2}}.$$

This system must have multiple spectra and therefore it must be noisy, in agreement with the theoretical and experimental results of Delande and others (Delande, 1991) and Iu and others (Iu, 1991).
(iv) Since it is hard to know whether or not a given $u(x, \tau)$ satisfies any of the infinite number of KdV equations, this $u_t - u_x = 0$, lowest in the hierarchy, enables us perhaps only qualitatively and semi-quantitatively to quantize a nonlinear nonintegrable two-degree-of-freedom system. As a preliminary finding, multiple and hence noisy quantum spectra are the manifestation of its deterministic chaos.

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