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ITERATIVE SOLUTION OF THE DIRICHLET PROBLEM FOR THE SEMILINEAR WAVE EQUATION

by

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ABSTRACT
Let Ω be an open bounded domain in $\mathbb{R}^n$. A solution to the Dirichlet problem for the semilinear wave equation

$$u_{tt} - \Delta u + g(t,x,u) = f(t,x) \text{ in } Q_T$$
$$u = 0 \text{ on } \Gamma = \partial\Omega \times [0,T]$$

where $Q_T = \Omega \times [0,T]$ is constructed using the equivalent abstract formulation

$$Lu + Nu = f$$

in the case where L and N satisfy monotonicity conditions. We also discuss an application to control theory.

1. INTRODUCTION

In this paper, we are interested in the existence, uniqueness, and, iterative approximation of solutions to the abstract operator equation

$$Lx + Nx = f \tag{1}$$

where L is a linear operator, N is a nonlinear operator, and f a fixed vector. It is known that several problems arising in mathematical physics can be suitably modelled, in the abstract, by (1). For instance, the one-dimensional Dirichlet problem for the semilinear wave equation

$$u_{tt} - u_{xx} + g(t,x,u) = f(t,x);\ (t,x) \in [0,T] \times (0,1) \tag{P1}$$
$$u(t,0) = u(t,1) = 0,$$

for all $t \in [0,T]$ can be put in the abstract form (1). (See, for example ref [1]). The n-dimensional case

$$u_{tt} - \Delta u + g(t,x,u) = f(t,x) \text{ in } Q_T$$
$$u = 0 \text{ on } \partial\Omega \tag{P2}$$
$$\partial u/\partial n = 0 \text{ on } \Gamma = \partial Q_T = [0,T] \times \partial\Omega$$

where $Q_T = [0,T] \times \Omega$, Ω an open bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $\partial u/\partial n$ is the outer normal derivative, can be treated in the same way. Further, such problems as

$$-\Delta u + g(x, \nabla u, \Delta u) + h(x,u) = f \text{ in } \Omega \tag{P3}$$
$$\partial u/\partial n = 0 \text{ on } \Gamma = \partial\Omega$$

can also be put in the abstract form (1). (See, for example, [2] and [3].) We remark immediately that the abstract Hammerstein equation

$$x + KNx = h \tag{2}$$

can be regarded as an equivalent form of (1), since on setting K =

L^{-1} (or $L = K^{-1}$), one obtains (2) from (1) (or (1) from (2)). This approach has been used by Chidume and Moore [4], and Moore [5] in their study of Hammerstein equations. It is known that every elliptic boundary value problem whose linear part possesses a Green's function can, as a rule, be transformed into the abstract form (2). Also, problems of the type

$$\begin{aligned} & x''' + ax'' + g(x') + cx = p(t); \; t \in [0, 2\pi] \\ & x(0) = x(2\pi) \\ & x'(0) = x'(2\pi) \\ & x''(0) = x''(2\pi) \end{aligned} \qquad \text{(P4)}$$

can be put into the form of the homogeneous Hammerstein equation

$$z + KNz = 0$$

(see, for example, [6].)

2. MAIN RESULTS

Let us recall that a Banach space X is called an upper-weak-parallelogram space with constant $b > 0$ (briefly, X is UWP(b)), in the terminology of Bynum [7], if for all x,y in X and $j \in Jx$ (where $J : X \longrightarrow 2^{X^*}$ is the normalized duality mapping and $\langle \cdot,\cdot \rangle$ the generalized duality pairing) we have that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j\rangle + b\|y\|^2 \qquad (3)$$

The L_p or l_p $(2 \leq p < \infty)$ are UWP(b) with b minorized by $p - 1$, that is, $b \geq p - 1$. We now prove the following result:

<u>Theorem 1</u>: Let X be UWP(b), $b \geq 1$. Suppose

(i) $L : X \longrightarrow X$ is a linear bounded and positive definite operator, that is, for each $x \in D(L)$, $\|Lx\| \leq k\|x\|$, some $k > 0$, and

$$\langle Lx, j\rangle \geq \alpha\|x\|^2; \text{ for some } \alpha > 0 \qquad (4)$$

(ii) $N : X \longrightarrow X$ is a nonlinear Lipschitz continuous bounded below operator, that is, for each pair x,y in D(N), $\|Nx - Ny\| \leq m\|x - y\|$, some $m > 0$ and

$$\langle Nx - Ny, w\rangle \geq -\beta\|x - y\|^2, \quad \beta \in \mathbb{R}; \; w \in J(x - y) \qquad (5)$$

If $\alpha - \beta = \lambda > 0$, then the abstract equation (1) has a unique solution for each $f \in X$ given. Moreover, this solution can be iteratively constructed using the usual Picard iterations. Further, convergence is at least as fast as a geometric progression with ratio

$$c = [1 - \lambda^2 b^{-1}(k + m)^{-2}]^{\frac{1}{2}} \in (0,1).$$

<u>Proof</u>: For each $x \in X$ and some constant $r > 0$, define the operator

$$T_r x = x - r(Lx + Nx - f).$$

Then,

$$\begin{aligned} \|T_r x - T_r y\|^2 &= \|x - y - r[(L + N)x - (L + N)y]\|^2 \\ &\leq \|x - y\|^2 - 2r\langle (L + N)x - (L + N)y, w\rangle \\ &\quad + r^2 b\|(L + N)x - (L + N)y\|^2 \\ &\leq [1 - 2r\lambda + r^2 b(k + m)^2]\|x - y\|^2 \\ &= [1 - \lambda^2 b^{-1}(k + m)^{-2}]\|x - y\|^2 \end{aligned}$$

(on setting $r = \lambda^2 b^{-1}(b + m)^{-2} \in (0,1)$.) So, for all $x,y \in X$,

$$\|T_r x - T_r y\| \leq c\|x - y\|$$

where

$$0 < c = ((1 - \lambda^2 b^{-1}(k + m)^{-2}))^{\frac{1}{2}} < 1.$$

Hence, T_r is strictly contractive and thus has a unique fixed point x^* in X, by the well-known Banach contraction mapping principle. Moreover, the successive approximations $x_0 \in X$ arbitrary, $x_{n+1} = T_r x_n$; $n \geq 0$ converge in norm to x^*, and the rate of convergence is given by

$$\|x_n - x^*\| \leq c^n \|x_0 - x^*\|$$

so that convergence is at least as fast as a geometric progression with ratio $c \in (0,1)$. Now, $x^* = T_r x^*$ iff $x^* = x^* - r((L + N)x^* - f)$ iff $Lx^* + Nx^* = f$. This completes the proof.

Remarks:

(1) If L is assumed to be nonlinear, Lipschitz continuous, and strongly accretive, the same conclusions are obtained.

(2) If L is bounded below with constant $-\beta$ and N is strongly accretive with constant $\alpha > 0$, such that $\alpha - \beta > 0$, we have the same conclusions.

(3) Condition (4) is often called the strong ellipticity condition for differential operators.

(4) The nature of the operators L and N often require that they map X into X^*, its dual, so that one uses monotonicity arguments, that is, one imposes monotonicity conditions. Thus, it becomes better to discuss such problems in the Hilbert space setting. Moreover, the weak formulations of (P1) to (P3) and their type show that the natural setting for their analysis are the Sobolev spaces $H^m(\overline{\Omega})$ which are Hilbert spaces, depending, of course, on the growth conditions satisfied by g and h. Since a Hilbert space is necessarily UWP(1), we have the following corollary to theorem 1:

Corollary 1: Let H be a Hilbert space. Suppose

(i) $L : H \longrightarrow H$ is a linear bounded strongly elliptic operator,

(ii) $N : H \longrightarrow H$ is a nonlinear Lipschitz continuous bounded-below operator with constant $-\beta$.

If $\alpha - \beta > 0$ then the conclusions of theorem 1 remain valid. The proof follows immediately on setting $b = 1$ and $J = I$ in theorem 1.

(5) The natural question to ask is, under what conditions will L and N satisfy the conditions of theorem 1 or its corollary? Now, if we set $Lu = u_{tt} - \Delta u$, then with appropriate boundary conditions, it is easily seen that L is linear, bounded, and strongly elliptic. Suppose that g grows like u, or g satisfies the growth conditions:

(i) $|g(t,x,u)| \leq \alpha(t,x) + \beta(t,x)|u|$, with $\alpha \in L_2(Q_T)$, and $\beta \in L_\infty(Q_T)$

(ii) $g(t,x,u)u \quad cu^2 - \delta(t,x)|u|$, with $c > 0$, $\delta \in L_2(Q_T)$.

Then, the Nemyckij operator $N : L_2 \longrightarrow L_2$ given by

$$Nu(.,.) = g(.,. u(.,.))$$

is monotone (accretive) or at worst, bounded-below and coercive, Lipschitz continuous or continuous and bounded, and, of course, nonlinear. Thus, the above results apply, to yield the unique solvability of the Dirichlet problem for the semilinear wave equation. If g grows like u^s, $0 < s < +\infty$, then the natural setting for the problem is the space L^p where $p = s + 1$. However, if g experiences exponential growth, that is, g grows like exp(u), then the natural setting is the so-called Orlicz space, which need not be reflexive. Since the Lipschitz continuity of N may not always be guaranteed, we seek a weaker continuity assumption on N that will work. Let us try hemicontinuity, which is often called continuity in the rays, that is, for each pair x,y in D(N), $N(x + t_n y) \rightharpoonup Nx$ as $t_n \longrightarrow 0^+$ where $\rightharpoonup$ denotes weak convergence. All linear and all continuous maps are hemicontinuous, as can easily be checked.

Let C be a bounded subset of a Hilbert space H and let $L : C \longrightarrow C$, $N : C \longrightarrow C$, and $S : C \longrightarrow H$ be such that $Sx = f + x - (L + N)x$. Then,

$$\|Sx - Sy\| \leq \|x - y\| + \|Lx - Ly\| + \|Nx - Ny\| \leq 3\,\delta(C) = 3\text{diam } C.$$

Hence,

$$\sup_{x\in C}\|Sx - Sy\| \leq 3\text{diam } C.$$

Thus, R(S) is bounded since C is bounded. We then have the following result:

Theorem 2: Let C be a bounded closed convex nonempty subset of H. Suppose

(i) $N : C \longrightarrow C$ is a hemicontinuous bounded-below (with constant $-\beta$) nonlinear operator,

(ii) $L : C \longrightarrow C$ is a linear strongly elliptic (with constant $\alpha > 0$) operator.

If $\alpha - \beta = \lambda > 0$, then (1) has a unique solution. Let $M = L + N$ and define $S : C \longrightarrow H$ by

$$Sx = f + x - Mx,\ x \in C$$

for some fixed f. Let $\{t_n\}$ be a real sequence such that

(i) $0 \leq t_n \leq 1$, for all $n \geq 0$

(ii) $\sum t_n = +\infty$

and

(iii) $\sum t_n^2 < +\infty$.

Then, the sequence $\{z_n\} \subset H$ recursively generated from arbitrary $x_0 \in C$ by

$$z_n = (1 - t_n)x_n + t_n Sx_n;\ n \geq 0$$

where $\{x_n\} \subseteq C$ is the sequence of points satisfying

$$\|x_{n+1} - z_n\| = \inf_{x\in C}\|x - z_n\|$$

that is, $x_{n+1} = R(z_n)$, converges strongly to the unique solution to (1), where $R : H \longrightarrow C$ is the proximity or retraction map. Moreover, if $t_n = 2\lambda^{-1}(n + 1)^{-1}$ for $n \geq n_0 > 0$, then the rate of convergence is of the form $O(n^{-\frac{1}{2}})$.

Proof: $\langle Mx - My, x - y\rangle \geq \lambda\|x - y\|^2$. So M is strongly monotone and hence monotone and coercive. L is linear and N is hemicontinuous, so M is hemicontinuous. Hence, the equation $Mx = Lx + Nx = f$ has a unique solution, say x^*, in C. Moreover, x^* is necessarily the unique fixed point of S. Also, $\langle Sx - Sy, x - y\rangle \leq (1 - \lambda)\|x - y\|^2$, so

$$\|z_n - x^*\|^2 = \|(1 - t_n)(x_n - x^*) + t_n(Sx_n - x^*)\|^2$$
$$\leq (1 - t_n)^2\|x_n - x^*\|^2 + $$
$$+ 2t_n(1 - t_n)\langle Sx_n - x^*, x_n - x^*\rangle$$
$$+ t_n^2\|Sx_n - x^*\|^2$$
$$\leq [(1 - t_n)^2 + 2(1 - \lambda)t_n(1 - t_n)]\|x_n - x^*\|^2$$
$$+ t_n^2 d^2$$
$$\leq (1 - \lambda t_n)\|x_n - x^*\|^2 + t_n^2 d^2$$

where

$$d = \sup_{n\geq 0}\|Sx_n - x^*\| < +\infty.$$

We then have, since R is nonexpansive in Hilbert spaces,

$$\|z_n - x^*\|^2 \leq (1 - \lambda t_n)\|z_{n-1} - x^*\|^2 + t_n^2 d^2$$

so that setting $\rho_{n+1} = \|z_n - x^*\|^2$ and $r_n = \lambda t_n$, we have,

$$\rho_{n+1} \leq (1 - r_n)\rho_n + t_n^2 d^2$$

from which we get, after induction, that

$$0 \leq \rho_n \leq Aw_n;\ n \geq 1$$

where $w_n \geq 0$ is recursively generated by

$$w_{n+1} = (1 - r_n)w_n + t_n^2;\ w_1 = 1$$

and

$$A = \max\{\rho_1, d^2\}.$$

We then have that $w_n \longrightarrow 0$ as $n \longrightarrow \infty$, and hence $\rho_n \longrightarrow 0$ as $n \longrightarrow \infty$, so that $z_n \longrightarrow x^*$ as $n \longrightarrow \infty$ as required. Now, for $n_o = (\lambda^{-1}(2 - \lambda)) + 1$, set $t_n = 2\lambda^{-1}(n + 1)^{-1}$. Then $\{t_n\}$ satisfies the requisite conditions for all $n \geq n_o$. From (6) we have

$$\rho_{n+1} \leq (1 - 2(n + 1)^{-1})\rho_n + (n + 1)^{-2}d_1^2.$$

Let

$$B = \max\{\rho_{n_o}, d_1^2\}.$$

Then we claim that

$$\rho_n \leq Bn^{-1} \text{ for all } n \geq n_o \tag{7}$$

Suppose that (7) is true for $n_o \leq n \leq k$. In particular, suppose

$$\rho_k \leq Bk^{-1}.$$

Then,

$$\rho_{k+1} \leq (1 - 2(k + 1)^{-1})\rho_k + (k + 1)^{-2}d_1^2$$
$$\leq (k - 1)(k + 1)^{-1}Bk^{-1} + B(k + 1)^{-2}$$
$$= B(k^2 + k - 1)(k + 1)^{-2}k^{-1} \leq B(k + 1)^{-1},$$

so that (7) is true for $n = k + 1$. Hence, by the inductive hypothesis, (7) is true for all $n \geq n_o$. Then, as required,

$$\| z_n - x^* \| = O(n^{-\frac{1}{2}}).$$

This completes the proof.

3. APPLICATION TO CONTROL THEORY

Let H_1 and H_2 be Hilbert spaces and let $X = L_2([0,T] : H_1)$ and $Y = L_2([0,T] : H_2)$ be the corresponding evolution (function) spaces, $0 \leq T \leq \infty$. (See, for example, [8]). Consider the semilinear control system

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t, x(t)) \\ x(0) &= x_o \end{aligned} \right\} \qquad (8)$$

where $-A : H_1 \longrightarrow H_2$ is a linear strongly elliptic operator with A (possibly) generating a C_o semigroup $S(t)$. $f : [0,T] \times H_1 \longrightarrow H_1$ is a nonlinear operator satisfying the Caratheodory condition (measurable in t for each $x \in H_1$ and continuous in x for each $t \in [0, T]$), and some suitable growth conditions; and $B : H_2 \longrightarrow H_1$ is a bounded linear operator. Y is the space of controls. Let us now assume that for any given control $u \in Y$ there exists a unique mild solution to (8) which can be expressed as

$$x(t) = S(t)x_o + \int_0^t S(T - s)Bu(s)ds + \int_0^t S(T - s)f(s,x(s))ds \qquad (9a)$$

Let us also define the operators $L : X \longrightarrow H_1$ and $N : X \longrightarrow H_1$ by

$$Lv = \int_0^T S(T - s)v(s)ds \qquad (9b)$$

$$Nv = \int_0^T S(T - s)F(Wv(s))ds \qquad (9c)$$

where $v \in \overline{R(B)}$, $F : X \longrightarrow X$ is the Nemyckij operator defined by $[Fx](t) = f(t,x(t))$, and $W : X \longrightarrow X$ is the solution operator defined as $W(v) = y$, where $y(t)$ is the unique mild solution to the system

$$\dot{y}(t) = Ay(t) + v(t) + f(t,y(t))$$
$$y(0) = x_o,$$

that is, W associates each given control with the corresponding mild solution. If the solution operator W is continuous, which is guaranteed by the strong ellipticity of $-A$ and the growth conditions on f, then N is continuous and bounded below or Lipschitz continuous. Also, N is at worst bounded below. Let $\overline{R(B)} = Z \subset X$ and consider the operators L and N as defined in (9) from Z into H_1. We now see that the conditions of theorem 2 and its corollary are satisfied so that the existence, uniqueness, and iterative approximation of the solution to $Lz + Nz = h$ are guaranteed. Hence, for each $h \in H_1$, there exists a $z \in Z$ such that

$$h = Lz + Nz = \int_0^T S(T - s)\left[z(s) + F(Wz(s))\right] ds \tag{10}$$

If we now define $Wz = y$, then we have that $h = y(T)$. Now, $z \in Z = \overline{R(B)}$ implies that for any $\varepsilon > 0$ given, there exists a control $u \in Y$ such that $\|z - Bu\|_Y \leq \varepsilon$. This immediately yields the controllability (or at worst approximate controllability) of the system (8). In particular, if B has a closed range, that is, $R(B) = \overline{R(B)}$, then we have that for each $z \in Z$ there exists a $u \in Y$ such that $Bu = z$ are elements of the evolution space X.

Example: Let us now look at the control system for the semilinear heat equation

$$\left.\begin{array}{l} z_t(t,x) = z_{xx}(t,x) + f(t,z(t,x)) + Bu(t,x); \\ 0 \leq t \leq T;\ 0 \leq x \leq \pi \\ z(t,0) = z(t,\pi) = 0;\ 0 \leq t \leq T \\ z(0,x) = 0;\ 0 \leq x \leq \pi \end{array}\right\} \tag{P5}$$

We make the assumption that for each $t \in [0,T]$, $u(t,x)$, and $z(t,x)$ belong to $L_2([0,\pi])$. Let $H = L_2([0,\pi])$ and $X = L_2([0,T]: H)$. We define the operator A to be $A \equiv -d^2(\ )/dx^2$ with domain

$$D(A) = \{v \in H : v'' \in H \text{ and } v(0) = v(\pi) = 0\}.$$

Then, (P5) transforms into the equivalent problem

$$\left.\begin{array}{l} \dot{v}(t) = -Av(t) + Fv(t) + Bu(t) \\ v(0) = 0 \end{array}\right\} \tag{P6}$$

where $v(t) = z(t,x)$ with $z(t,0) = z(t,\pi) = 0$ for all $0 \leq t \leq T$ and $Fv(t) = f(t,v(t))$. As can easily be checked, $-A$ is strongly elliptic and A generates a compact semigroup $S(t)$. Suitable growth conditions on f (for instance if f grows like z) ensure that the Nemyckij operator F and hence N possess the requisite properties that yield the existence, uniqueness, and iterative constructibility of the corresponding abstract equation $Lz + Nz = h$, and hence the controllability of the system.

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