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AN ERROR ESTIMATE FOR TRÉMOLIÈRES' METHOD FOR THE DISCRETIZATION OF PARABOLIC VARIATIONAL INEQUALITIES

by

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ABSTRACT
We study a scheme for the time-discretization of parabolic variational inequalities that is often easier to use than the classical method of Rothe. We show that if the data are compatible in a certain sense, then this scheme is of order greater than or equal to $\frac{1}{2}$.

1. INTRODUCTION

Let V and H be Hilbert spaces satisfying $V \subset H = H' \subset V'$, with continuous and dense inclusions. We use $(.,.)$ to denote both the inner product of H (with associated norm $|.|$) and the duality between V' and V, while $\|.\|$ will denote the norm of V. Let $A : V \to V'$ be a linear function such that $\exists\ M > 0,\ c > 0$ such that

$$(Av,v) \geq c\|v\|^2 \ \forall v \in V \tag{1}$$

$$|(Av,z)| \leq M\|v\|\|z\| \ \forall v, z \in V \tag{2}$$

Given a closed convex subset K of V, we consider the following problem:

$$\left.\begin{array}{l} u(0) = u_0 \\ u(t) \in K \\ (u'(t) + Au(t) - f(t), u(t) - v)) \leq 0 \ \forall v \in K,\ \forall t \in (0,T) \end{array}\right\} \tag{3}$$

Naturally, the existence and the properties of the solution to this problem depend on the function space in which the solution is sought and on the data f and u_0. Results on these questions can be found in [8,2,7], while the numerical solution of the problem is studied in [6,9,10,12]. See also the numerous references cited in these works. One of the simplest methods for discretizing this problem is the method of Rothe (cf [3]):

$$\left.\begin{array}{l} \tilde{u}_k^0 = u_0 \\ \tilde{u}_k^n \in K \\ ((\tilde{u}_k^n - \tilde{u}_k^{n-1})k^{-1} + A\tilde{u}_k^n, \tilde{u}_k^n - x) \\ \qquad \leq (f_k^n, \tilde{u}_k^n - x) \ \forall x \in K,\ n = 1,\ldots,N \end{array}\right\} \tag{4}$$

where $k = T/N$ and

$$f_k^n = (1/k)\int_{(n-1)k}^{nk} f(t)dt.$$

Let $\tilde{u}_k$ denote the step function whose value on the interval $((n-1)k, nk)$ is $\tilde{u}_k^n$ (we will adopt an analogous notation for other functions Pu_k, f_k, etc) and let $\mathbb{K}$ denote the closure of K in H. A result due to Baiocchi [3] states that if $u_0 \in \mathbb{K}$ and $f \in L_2(0,T,H)$, then

$$|\tilde{u}_k - u|^2_{L_\infty(0,T,H)} + \|\tilde{u}_k - u\|^2_{L_2(0,T,V)} = O(k) \tag{5}$$

In the same paper, it is also shown that if $u_0 \in K$, $f \in L_2(0,T,V)$, $f' \in L_2(0,T,V')$ and $Au_0 - f(0) \in H$, then we have the first order error estimate:

$$|\tilde{u}_k - u|^2_{L_\infty(0,T,H)} + \|\tilde{u}_k - u\|^2_{L_2(0,T,V)} = O(k^2) \tag{6}$$

The semi-discrete problems (4) are elliptic variational inequalities possessing unique solutions [4,11]. But it is not often clear how to obtain the numerical solutions of these elliptic problems. If P denotes the orthogonal projection from H to $\mathbb{K}$, then we discretize eq (3) using the alternative method $u_k^0 = u_0$,

$$u_k^n - Pu_k^{n-1} + kAu_k^n = kf_k^n,\ n = 1,\ldots,N \tag{7}$$

This method is clearly easier to use than eq (4) whenever the projection P is explicitly known (this occurs, for instance, for obstacle problems: cf [4]). The semi-discrete problems (7) are elliptic equations which can be discretized using any of the standard methods for space discretization (see, for example, [1,13]) and solved using the standard methods for solving systems of linear equations. This scheme generalizes a method used in [14] (it is referred to in [9] as the "implicit in equation" scheme). For this reason, we call it the method of Trémolières. The main purpose of this paper is to prove an error estimate analogous to (5) for this method. We consider only time discretization. However, our estimates can be combined with known error estimates for the space discretization [1,14] of elliptic equations to obtain error estimates for the simultaneous space and time discretization. The simultaneous space and time discretization - with error estimates - of some parabolic variational inequalities has been studied in [6,10]. [10] solves a problem of the type (3) using the method of Rothe for the time discretization and the finite element method for the discretization in space, while [6] solves a problem of the same type using a finite element space discretization and time discretization of the form (5) with $A\tilde{u}_k^n$ replaced by $A\tilde{u}_k^{n-1}$.

2. RESULTS

We assume that

$$\left.\begin{array}{l} v \in V \Longrightarrow Pv \in V, \text{ and } \exists\, w \in H \text{ such that} \\ v \in V \Longrightarrow (v - Pv,\ APv - w) \geq 0 \end{array}\right\} \tag{8}$$

This hypothesis holds in many applications. For example, let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary, $H = L_2(\Omega)$, $V =$

$H_0^1(\Omega)$ and let $\psi \in H^2(\Omega)$ satisfy $\psi|_{\partial\Omega} \leq 0$ (in the sense of traces). If we set $A = -\Delta$ and $K = \{v \in V : v \geq \psi\}$, then (8) holds with $w = \Delta\psi$. In fact, we have (cf [11,5]),

$$Pv = v + (\psi - v)^+.$$

If $v \in V$, then since $v|_{\partial\Omega} = 0$ (in the sense of traces) and $\psi|_{\partial\Omega} \leq 0$, we have $(\psi - v)^+ \in V$. Therefore, $Pv \in V$. Also, it follows from [11, theorem A1] that

$$\nabla(\psi - v)^+(x) = \begin{cases} \nabla\psi(x) - \nabla v(x) & \text{in } \{x \in \Omega : \psi(x) - v(x) > 0\} \\ 0 & \text{in } \{x \in \Omega : \psi(x) - v(x) \leq 0\} \end{cases}$$

Therefore, we have

$$\begin{aligned}(Pv - v, APv - w) &= \int_\Omega \nabla(\psi - v)^+(\nabla v - \nabla\psi + \nabla(\psi - v)^+)dx \\ &= \int_{\psi(x)-v(x)>0} \nabla(\psi - v)(\nabla v - \nabla\psi + \nabla(\psi - v))dx \\ &= 0.\end{aligned}$$

Therefore, (8) holds, in this case. We now begin the study of the scheme (7). In the sequel, G will designate a series of constants, not having the same value in any two places.

<u>Theorem 1</u>: If $f \in L_2(0,T,H)$ then

$$|u_k - Pu_k|^2_{L_2(0,T,H)} \leq Gk^2(|f|^2_{L_2(0,T,H)} + |w|^2) \tag{9}$$

$$|u_k - Pu_k|^2_{L_\infty(0,T,H)} + \|u_k - Pu_k\|^2_{L_2(0,T,V)} \leq Gk(|f|^2_{L_2(0,T,H)} + |w|^2) \tag{10}$$

<u>Proof</u>: Taking the scalar product with $u_k^n - Pu_k^n$ in (7), we obtain

$$(Pu_k^n - Pu_k^{n-1}, u_k^n - Pu_k^n) + k(Au_k^n - w, u_k^n - Pu_k^n) + |u_k^n - Pu_k^n|^2 \leq k(f_k^n - w, u_k^n - Pu_k^n).$$

Using the definition of P, we obtain

$$k(Au_k^n - w, u_k^n - Pu_k^n) + |u_k^n - Pu_k^n|^2 \leq k(f_k^n - w, u_k^n - Pu_k^n).$$

If we rewrite this in the form

$$k(APu_k^n - w, u_k^n - Pu_k^n) + k(Au_k^n - APu_k^n, u_k^n - Pu_k^n) + |u_k^n - Pu_k^n|^2 \leq k(f_k^n - w, u_k^n - Pu_k^n)$$

and use (8), we obtain

$$k(Au_k^n - APu_k^n, u_k^n - Pu_k^n) + |u_k^n - Pu_k^n|^2 \leq k(f_k^n - w, u_k^n - Pu_k^n).$$

This yields

$$\begin{aligned}kc\|u_k^n - Pu_k^n\|^2 + \tfrac{1}{2}|u_k^n - Pu_k^n|^2 &\leq \tfrac{1}{2}k^2|f_k^n - w|^2 \\ &\leq k^2|f_k^n|^2 + k^2|w|^2.\end{aligned}$$

From this, we easily obtain (9) and (10). We now give the error estimate of the approximation (7).

Theorem 2: If $u_0 \in \mathbb{K}$ and $f \in L_2(0,T,H)$, then there exists $u \in L_2(0,T,V)$ such that u solves (3) and

$$Pu_k \longrightarrow u \text{ in } L_\infty(0,T,H) \quad (11)$$

$$u_k \longrightarrow u \text{ in } L_2(0,T,V) \quad (12)$$

Furthermore, we have the estimate

$$|Pu_k - u|^2_{L_\infty(0,T,H)} + \|u_k - u\|^2_{L_2(0,T,V)} = O(k) \quad (13)$$

Proof: The proof is based on the known properties of the Euler approximation (4). It follows from the results of [3] that (3) has a unique solution u to which $\tilde{u}_k$ converges strongly in the norm $|\cdot|_{L_\infty(0,T,H)} + \|\cdot\|_{L_2(0,T,V)}$. From (7) and (4), we have, respectively,

$$((Pu_k^n - Pu_k^{n-1})/k + Au_k^n - f_k^n, Pu_k^n - \tilde{u}_k^n)$$
$$+ ((u_k^n - Pu_k^n)/k, Pu_k^n - \tilde{u}_k^n) \leq 0$$

and

$$((\tilde{u}_k^n - \tilde{u}_k^{n-1})/k + A\tilde{u}_k^n, \tilde{u}_k^n - Pu_k^n) \leq (f_k^n, \tilde{u}_k^n - Pu_k^n).$$

Adding these inequalities, we obtain

$$(Pu_k^n - \tilde{u}_k^n + kAu_k^n - kA\tilde{u}_k^n, Pu_k^n - \tilde{u}_k^n)$$
$$\leq (Pu_k^{n-1} - \tilde{u}_k^{n-1}, Pu_k^n - \tilde{u}_k^n).$$

This implies

$$\tfrac{1}{2}|Pu_k^n - \tilde{u}_k^n|^2 - \tfrac{1}{2}|Pu_k^{n-1} - \tilde{u}_k^{n-1}|^2 + kc\|u_k^n - \tilde{u}_k^n\|^2$$
$$\leq k(Au_k^n - A\tilde{u}_k^n, u_k^n - Pu_k^n).$$

Hence

$$\tfrac{1}{2}|Pu_k^n - \tilde{u}_k^n|^2 - \tfrac{1}{2}|Pu_k^{n-1} - \tilde{u}_k^{n-1}|^2 + kc\|u_k^n - \tilde{u}_k^n\|^2$$
$$\leq Mk\|u_k^n - Pu_k^n\|\,\|u_k^n - \tilde{u}_k^n\|.$$

Therefore, using (10), we obtain

$$\tfrac{1}{2}|Pu_k^m - \tilde{u}_k^m|^2 + \tfrac{1}{2}kc\sum_{n=1}^{m}\|u_k^n - \tilde{u}_k^n\|^2$$
$$\leq (kM^2/2c)\sum_{n=1}^{N}\|u_k^n - Pu_k^n\|^2$$
$$\leq Gk(|f|^2_{L_2(0,T,H)} + |w|^2) \;\forall\, 1 \leq m \leq N.$$

This inequality, and the error estimate (5), imply that

$$|Pu_k - u|^2_{L_\infty(0,T,H)} + \|u_k - u\|^2_{L_2(0,T,V)}$$
$$\leq 2|Pu_k - \tilde{u}_k|^2_{L_\infty(0,T,H)} + 2|\tilde{u}_k - u|^2_{L_\infty(0,T,H)}$$

$$+ 2 \|u_k - \tilde{u}_k\|^2_{L_2(0,T,V)}$$

$$+ 2 \|\tilde{u}_k - u\|^2_{L_2(0,T,V)} = O(k).$$

Equations (11) and (12) follow easily from this. In conclusion, we remark that the regularity assumption we made, namely $f \in L_2(0,T,H)$ and $u_0 \in \mathbb{K}$, yielded an error estimate which is analogous to the estimate (3), which holds for the method of Rothe under the same conditions. We do not know if the stronger conditions $u_0 \in K$, $f \in L_2(0,T,H)$, $f' \in L_2(0,T,V')$, and $f(0) - Au_0 \in H$ will yield an error estimate of order one, as is the case with the method of Rothe. In [15], a more general variational inequality has been studied, using a discretization method that generalizes (7). However, the compatibility condition used in that paper does not reduce to (8) as a special case.

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