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ITERATIVE SOLUTION OF CERTAIN NONLINEAR OPERATOR EQUATIONS ARISING IN MATHEMATICAL PHYSICS

by

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ABSTRACT

Let Ω be an open domain bounded in $\mathbb{R}^n$ and let L be some differential operator. The mixed initial-boundary value problem:

$$u_t + Lu = f(x,u) \text{ in } \Omega \times (0,T)$$

$$Bu = g(x,u) \text{ on } \Gamma = \partial\Omega \times (0,T)$$

$$u(0,x) = u_0(x) \text{ in } \Omega,$$

is often a suitable mathematical model for several situations arising in chemical flows, gas dynamics, heat conduction, and other physical processes. Using a purely abstract approach, the existence, uniqueness, and strong convergence of fixed point iterations to a solution to the above problem is established.

1. INTRODUCTION

Time-dependent irreversible processes such as heat conduction, diffusion, chemical reactions, biological processes, etc, are frequently modelled mathematically by semi-linear parabolic differential equations of the form

$$u_t + Lu = f(x,u) \text{ in } \Omega \qquad \text{(P1)}$$

$$Bu = 0 \text{ on } \Gamma = \partial\Omega$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with a smooth boundary Γ and $u = u(x,t)$. The nonlinear term f represents the interactions of the process. The solutions to the corresponding semi-linear elliptic boundary value problems

$$\left.\begin{aligned} Lu &= f(x,u) \text{ in } \Omega \\ Bu &= g(x,u) \text{ on } \Gamma \end{aligned}\right\} \qquad \text{(P2)}$$

represent stationary states. (We observe that (P2) is an example of the so-called Steklov problem, since $g = g(x,u)$). Typical examples of (P2) are the cases

(i) $Lu = -\Delta u + a(x,u)$

$Bu = u$

(ii) $Lu = -\Delta u + a(x)u$

$Bu = \partial u/\partial n + \beta(x)u$

and

(iii) $Lu = -\sum_{i=1}^{N} a_{ij} D_i D_j u + \sum_{i=1}^{N} a_i D_i u + au$

$Bu = u + \partial u/\partial n$

where $\partial u/\partial n$ denotes the outer normal derivative. Here $u = u(x)$, that is, u is time independent.

Let us consider a modified form of (P1):

$$u_t + Lu = f(x,u) \text{ in } Q_T$$
$$Bu = 0 \text{ on } \Gamma \qquad (P3)$$
$$u(x,0) = u_o(x)$$

where $Q_T = \Omega \times (0,T)$, $T > 0$, and $\Gamma = \partial\Omega \times (0,T)$. (P2) with g = 0 is the corresponding stationary elliptic problem. If u is taken to be the temperature of the body Ω, then (P3) means that we are given an initial temperature at time t = 0 and the behaviour of the temperature on the boundary $\partial\Omega$ for the time period (0,T), that is, either

$$u = 0$$

or

$$\beta u + \partial u/\partial n = 0$$

on $\partial\Omega$ for all $t \in (0,T)$. (Suppose $v > 0$ denotes the heat conductivity of the body Ω. Then, $j = -v\,\partial u/\partial n$ is the heat flux density in the direction of the outward normal. If u = 0 is the external temperature, then $\beta \geq 0$ necessarily.) It is known, from observation, that stable processes reach a stationary final state of temperature u where u is no longer time dependent, and hence we have (P2). This evidence leads to the important main stability question: Which initial states evolve into stable final states as t goes to infinity? The accepted way of answering the above question is to work in ordered Banach spaces using order cones, and to regard initial states as sub- or super-solutions to (P2), the stationary problem, and the final states as the corresponding smallest or greatest solution to (P2). Then, using such known results as that if w is a subsolution and v is a supersolution, then there exists $u^* \in [w,v]$ such that u^* is a solution. Details of our efforts in this area will be presented in a future publication. The interested reader may, however, consult Deimling [1] and Zeidler [2], and the references cited therein. Here, we adopt a purely abstract approach.

To ensure monotonicity of the right hand side of (P2), we carry out the following modification

$$u_t + Lu + ru = ru + f(x,u) \text{ in } Q_T$$
$$Bu = 0 \text{ on } \Gamma \qquad (P4)$$
$$u(x,0) = u_o(x)$$

Notice that the Green's function approach transforms (P4) or (P3) to the equivalent abstract formulation

$$u + KNu = 0 \qquad (1)$$

where K is the linear integral operator on the Green's function as its kernel, while N is the Nemyckij operator:

$$Nu(x,t) = ru(x,t) + f(x,u(x,t)) \qquad (2)$$

In spite of our abstract approach, the conditions we posit are governed by their relevance to physical processes. For example, K is often strongly elliptic, that is, $\langle Ku,u\rangle \geq c\|u\|^2$; $c > 0$, and f satisfies suitable growth conditions which ensure that the Nemyckij operator satisfies certain continuity and monotonicity conditions.

2. PRELIMINARIES

Let V be a normed linear space and V^* its dual. Let $J: V \longrightarrow 2^{V^*}$ be the normalized duality mapping and $\langle .,. \rangle$ the generalized duality pairing between V and V*. Suppose $T : V \longrightarrow V^*$ satisfies the conditions that there exists a mapping $F : V^* \longrightarrow V$ (V^{**}) with $R(T) \subseteq D(F)$ such that for each pair x,y in D(T) and $w \in J(x-y)$ we have

$$\mathrm{Re} \langle Tx - Ty, Fw \rangle \geq -\lambda \|x - y\|^2; \lambda \in \mathbb{R} \tag{3}$$

Then T is called F-bounded below with constant λ. If $T : V \longrightarrow V$ then $F : V \longrightarrow V$ with $D(T) \subseteq D(F)$ and $z \in J(Fx - Fy)$, (3) becomes

$$\mathrm{Re} \langle Tx - Ty, z \rangle \geq -\lambda \|x - y\|^2 \tag{4}$$

For a linear T and F with $w \in Jx$, $z \in J(Fx)$, we have

$$\mathrm{Re} \langle Tx, Fw \rangle \geq -\lambda \|x\|^2 \tag{3a}$$

and

$$\mathrm{Re} \langle Tx, z \rangle \geq -\lambda \|x\|^2 \tag{4a}$$

Several classes of operators such as the classes of K*-positive-definite, positive, bounded below, and invertible operators are subclasses of this class of operators. See, for example, Petryshyn [3], Chidume and Aneke [4], and Moore [5]. We now give an example to show that such operators as defined above do actually exist.

<u>Example</u>: Consider the operator

$$Ax = x'' + x' - x(x^3).$$

It is straightforward to see that A satisfies the following condition on a Hilbert space over some given interval $I = [a,b]$ where $x(a) = x(b) = 0$:

$$\mathrm{Re} \langle Ax, x \rangle \leq -\|x\|^2$$

or, which is the same thing,

$$\mathrm{Re} \langle Ax - Ay, x - y \rangle \leq -\|x - y\|^2,$$

so that A is dissipative. But if we define

$$Kx = -2cx; \; c > 1,$$

then we have that

$$\mathrm{Re} \langle Ax, Kx \rangle \geq 2c\|x\|^2 = (2c)^{-1}\|Kx\|^2$$

or

$$\mathrm{Re} \langle Ax - Ay, Kx - Ky \rangle \geq 2c\|x - y\|^2 = (2c)^{-1}\|Kx - Ky\|^2.$$

Thus A is K-positive-definite (K-strongly accretive) but it is not positive (accretive or monotone). If $Kx = cx$, $c > 0$, then

$$\mathrm{Re} \langle Ax, Kx \rangle \geq -c\|x\|^2$$

or

$$\mathrm{Re} \langle Ax - Ay, Kx - Ky \rangle \geq -c\|x - y\|^2.$$

3. MAIN RESULTS

<u>Theorem 1</u>: Let X be a real reflexive Banach space and X* its dual. Suppose $K : X^* \longrightarrow X$ is a bounded linear operator and $N : X \longrightarrow X^*$ is a hemicontinuous K*-bounded-below operator with constant $\lambda \in (-\infty, 1)$. Then for each fixed $f \in X$, the Hammerstein equation

$$x + KNx = f \tag{5}$$

has a unique solution. (K* is the adjoint or conjugate of K.)

<u>Proof</u>: Let $S = I + KN$. Then,

$$\begin{aligned} \langle Sx - Sy, w \rangle &= \langle x - y, w \rangle + \langle KNx - KNy, w \rangle \\ &= \langle x - y, w \rangle + \langle Nx - Ny, K^*w \rangle \\ &\geq (1 - \lambda)\|x - y\|^2. \end{aligned}$$

Hence, the Hammerstein operator $S = I + KN$ is strongly monotonic with constant $1 - \lambda > 0$. Observe that S is hemicontinuous since N is, by hypothesis, K is linear, and I is the identity operator. Moreover, setting $y = 0$, we have

$$\langle Sx, w\rangle \geq (1 - \lambda)\|x\|^2 + \langle KNo, w\rangle$$

so that

$$\langle Sx, w\rangle / \|x\| > (1 - \lambda)\|x\| + \langle So, w\rangle / \|x\|$$

as $x \longrightarrow \infty$. Hence, S is coercive. Thus $R(S) = X$ since S is monotone, hemicontinuous, and coercive. Therefore (5) is solvable for each given $f \in X$ fixed.

Suppose u and z are solutions to (5), that is, $Su = f = Sz$. Then, with $w \in J(u - z)$, we have

$$0 = \langle Su - Sz, w\rangle \geq (1 - \lambda)\|u - x\|^2.$$

Hence, $u = z$ and the solution to (5) is necessarily unique. This completes the proof.

<u>Remark</u>: Let V be a normed linear space. Suppose $N : V \longrightarrow V^*$ is everywhere defined in V, that is, $D(N) = V$. Then, $S = I + KN : V \longrightarrow V$ is also everywhere defined in V, that is, $D(S) = V$. (Of course, $K : V^* \longrightarrow V$ and $I : V \longrightarrow V$). Now, if N is K*-bounded-below with constant $\lambda < 1$, then S is strongly accretive (monotonic) with constant $1 - \lambda > 0$. Thus

$$\|Sx - Sy\| \, \|x - y\| \quad \langle Sx - Sy, w\rangle \geq (1 - \lambda)\|x - y\|^2.$$

Hence

$$\|Sx - Sy\| \geq (1 - \lambda)\|x - y\|.$$

S is, therefore, injective, and so, uniquely invertible. Observe that $S^{-1} \in Lip(k)$, $k = (1 - \lambda)^{-1}$. Now, since S is injective and $D(S) = V$, then S is necessarily surjective, that is, $R(S) = V$. Thus, for each fixed $f \in V$, $q = S^{-1}f$ is the unique solution to (5).

<u>Corollary 1</u>: Let V be a normed linear space and V* its dual. Let $K : V^* \longrightarrow V$ be a bounded linear operator and $N : V \longrightarrow V^*$ be a nonlinear-everywhere-defined K*-bounded-below with constant $\lambda < 1$ operator. Then, for each fixed $f \in V$ given, the Hammerstein equation (5) has a unique solution.

The results above are topological. The next result is both topological and algebraic.

<u>Theorem 2</u>: Let X be UWP(b), $b \geq 1$. Let $m > 0$ and K, N be as in theorem 1, with $N \in Lip(m)$. Set $L \equiv 1 + \|K\|m$ and assume that

$$0 < 1 - (1 - \lambda)^2/bL^2 < 1.$$

Then, (5) has a unique solution. Moreover, the Picard iterations for (5) converge in norm to this unique solution at least as fast as a geometric progression with ratio

$$c = (1 - (1 - \lambda)^2/bL^2)^{\frac{1}{2}}.$$

<u>Proof</u>: $N \in Lip(m)$. Then,

$$\|Sx - Sy\| = \|x + KNx - y - KNy\|$$
$$\leq \|x - y\| + \|K\|\|Nx - Ny\|$$
$$\leq (1 + \|K\|m)\|x - y\| = L\|x - y\|.$$

Let us define the auxilliary fixed point operator

$$T_r x = x - r(Sx - f).$$

Obviously, $T_r x^* = x^*$ iff $Sx^* = f$. Now,

$$\|T_r x - T_r y\|^2 = \|(x - y) - r(Sx - Sy)\|^2$$
$$\leq \|x - y\|^2 - 2r \quad Sx - Sy, w \quad + r^2 b\|Sx - Sy\|^2$$
$$\leq [1 - 2(1 - \quad)r + bL^2r^2]\|x - y\|^2$$
$$= [1 - (1 - \quad)^2/bL^2]\|x - y\|^2,$$

on setting $0 < r = (1 - \lambda)/bL^2 < 1$. Hence,

$$\|T_r x - T_r y\| \leq c\|x - y\|$$

where

$$c = (1 - (1 - \lambda)^2/bL^2)^{\frac{1}{2}} \in (0,1),$$

by hypothesis. Thus, T_r is a strict contraction and hence has a unique fixed point which is the unique solution to (5). For $x_o \in X$ arbitrary, let

$$x_{n+1} = T_r x_n, \; n \geq 0.$$

Then, we have, by the Banach contraction mapping theorem, that $x \longrightarrow x^*$ as $n \longrightarrow \infty$, and

$$\|x_n - x^*\| \leq c^n\|x_o - x^*\|$$

or

$$\|x_n - x^*\| \leq c^n/(1 - c)\|x_1 - x_o\|.$$

This completes the proof.

Remark: A solution to (P3) is a function $u \in C([0,T]: C(\bar{\Omega}))$. Now, since $\Omega \subseteq \mathbb{R}^n$ is bounded, then $\bar{\Omega}$ is compact. Also, $[0,T]$ is a compact interval of $\mathbb{R}$. Hence, since every continuous function on a compact set is bounded on the set, we have the following continuous embeddings:

$$C([0,T] : C(\bar{\Omega})) \hookrightarrow L^{\infty}([0,T] : L^{\infty}(\bar{\Omega}))$$
$$\hookrightarrow L^p([0,T] : L^p(\bar{\Omega}))$$

$(1 \leq p < \infty)$. Hence, we work in the L^p spaces. Observe that L^p, $2 \leq p < \infty$, spaces are UWP(b) with b minorized by p - 1, that is, $b \geq p - 1$.

Theorem 3: Let X, K, and N be as in theorem 2. Define $T : X \longrightarrow X$ by $Tx = f - KNx$. Let $\{t_n\}$ be a real sequence satisfying

(i) $0 \leq t_n \leq (1 - \lambda)/(1 + bL^2 - 2\lambda) < 1$; $n \geq 0$

(ii) $\sum t_n = +\infty$.

Then, the iterative sequence generated from $x_o \in X$ arbitrary,

$$x_{n+1} = (1 - t_n)x_n + t_n Tx_n; \; n \geq 0$$

converges strongly to the unique solution to (5). Moreover, if $t_n = (1 - \lambda)/(bL^2 - 2\lambda + 1)$, then the convergence rate is at least as fast as a geometric progression with ratio $(1 - \mu)^{\frac{1}{2}}$,

$$\mu = (1 - \lambda)^2/(bL^2 - 2\lambda + 1) \in (0,1),$$

where L is the Lipschitz constant of KN and $\lambda^2 < bL^2$.

Proof: $Tx^* = x^*$ if and only if $x^* + KNx^* = f$. Also, for each pair x,y in X,

$$\langle Tx - Ty, w\rangle = -\langle KNx - KNy, w\rangle$$
$$= -\langle Nx - Ny, K^*w\rangle$$
$$\leq \lambda\|x - y\|^2.$$

Set $\rho_n = \|x_n - x^*\|^2$; $w_n \in J(x_n - x^*)$ to get

$$\rho_{n+1} \leq [(1-t_n)^2 + 2\ t_n(1-t_n) + bL^2t_n^2]\rho_n$$
$$\leq [1-(1-\ \ )t_n]\rho_n \qquad \text{(using condition (i))}$$
$$\leq [\exp(1-(1-\lambda)t_n)]\rho_n.$$

Thus,

$$\rho_{m+1} \leq [\exp(1-(1-\lambda)\sum_{0}^{m} t_n)]\rho_0,$$

which goes to zero as m goes to infinity, by condition (ii). Hence, $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. If $t_n = (1-\lambda)/(bL^2 - 2\lambda + 1)$, then,

$$\rho_{n+1} \leq [1-(1-\lambda)^2/(bL^2 - 2\lambda + 1)]\rho_n$$

and the result follows, completing the proof.

Remark: In (3) and (4), if $\lambda < 0$ so that $\alpha = -\lambda > 0$, then T is said to be F-positive definite. Or, to distinguish this class of operators from the class studied by Petryshyn [3], we say that this T is F-positive bounded below. We thus have the following corollary on noting that we may take $\alpha \in (0,1)$ without loss of generality.

Corollary 2: In theorem 3, let N be K*-positive-definite with constant $\alpha \in (0,1)$ and let the real sequence $\{t_n\}$ satisfy, in place of condition (i),

(i)' $\quad 0 \leq t_n \leq (1+\alpha)/(bL^2 + 2\alpha + 1);\ n \geq 0.$

Then, the same conclusions are obtained with

$$\mu = (1+\alpha)^2/(bL^2 + 2\alpha + 1)$$

if

$$t_n \equiv (1+\alpha)/(bL^2 + 2\alpha + 1).$$

Proof: Set $\alpha = -\lambda$ in theorem 2 and the result follows.

Example: Consider the 2-dimensional elliptic problem

$$\partial(pu_x)/\partial x + \partial(pu_y)/\partial y = w(x,y;\ u(x,y)) \qquad (6)$$

with $p(x,y) > 0$ and prescribed appropriate boundary conditions so that the linear part possesses a Green's function $k(x,y;r,s)$. Obviously, (6) has the equivalent formulation

$$u_{xx} + u_{yy} = g(x,y;\ u(x,y)) - h(x,y;\ u(x,y))$$

where $g = p^{-1}w$ and $h = p^{-1}(p_x u_x + p_y u_y)$. We then obtain

$$u(x,y) = -\int_{\Omega} k(x,y;r,s)f(r,s;\ u(r,s))drds$$

where Ω is a bounded region in $\mathbb{R}^2$ and $f = g - h$. So that defining the linear integral operator

$$Kv(\cdot,\cdot) = \int_{\Omega} k(x,y;\ \cdot,\cdot)v(\cdot,\cdot)drds$$

and the Nemyckij operator

$$Nu(\cdot,\cdot) = f(\cdot,\cdot;\ u(\cdot,\cdot))$$

we then have the abstract form

$$u + KNu = 0$$

which is a homogeneous Hammerstein equation. Let us now state and prove theorem 4.

Theorem 4: Let E be a Banach space with a uniformly convex dual E*. Let $K : E^* \longrightarrow E$ be bounded linear and $N : E \longrightarrow E^*$ be K*-bounded below. Let C be a symmetric bounded closed convex subset of E (for example, $C = \overline{B}(0,d)$, $d < \infty$) and suppose that $K : N(C) \longrightarrow C$, that is, K maps the image of C under N into C. Define $T : C \longrightarrow C$ by $Tx = -KNx$. Let $\{t_n\}$ be a real sequence satisfying the conditions

(i) $0 \leq t_n \leq 1$, all $n \geq 0$,

(ii) $\sum t_n = \infty$,

and

(iii) $\sum t_n b(t_n) < +\infty$.

Then, the iterative sequence $\{x_n\}$ converges in norm to the unique solution to (7).

Proof:

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1 - t_n)(x_n - x^*) + t_n(Tx_n - x^*)\|^2 \\ &\leq (1 - t_n)^2\|x_n - x^*\|^2 \\ &\quad + 2t_n(1 - t_n)\langle Tx_n - x^*, j(x_n - x^*)\rangle \\ &\quad + \max[(1 - t_n) x_n - x^* , 1] t_n\|Tx_n - x^*\| \cdot \\ &\qquad \cdot b(t_n\|Tx_n - x^*\|) \\ &\leq (1 - t_n)^2\|x_n - x^*\|^2 \\ &\quad + 2t_n(1 - t_n)\langle Tx_n - x^*, j(x_n - x^*)\rangle \\ &\quad + \max[(1 - t_n)\|x_n - x^*\|, 1] \cdot \\ &\qquad \cdot \max[\|Tx_n - x^*\|, 1]\|Tx_n - x^*\|t_n b(t_n) \\ &\leq [1 - (1 - \lambda)t_n]\|x_n - x^*\|^2 + Mt_n b(t_n)\end{aligned}$$

and hence,

$$\rho_{n+1} \leq (1 - r_n)\rho_n + M\sigma_n$$

where $r_n = (1 - \lambda)t_n$, $\rho_n = \|x_n - x^*\|^2$, and $\sigma_n = t_n b(t_n)$. Well-known arguments (see, for example, ref [6]) now show that $\rho_n \longrightarrow 0$ as $n \longrightarrow \infty$, that is, $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. This completes the proof.

Remarks: 1. If N is Lipschitz continuous, then the requirement that K maps the image of C under N into C can be dispensed with.

2. If $t_n = s/(\lambda(n + 1))$; $1 < s \leq 2$, then a convergence rate of the order $n^{-\frac{1}{2}(s - 1)}$ is obtained. That is,

$$\|x_n - x^*\| = O(n^{-\frac{1}{2}(s - 1)}).$$

Hence, for $E = L_p$; $1 < p < \infty$,

$$\|x_n - x^*\| = \begin{cases} O(n^{-\frac{1}{2}(p - 1)}); & \text{if } 1 < p \leq 2 \\ O(n^{-\frac{1}{2}}); & \text{if } 2 \leq p < \infty. \end{cases}$$

Corollary 3: In theorem 4, let $E = L_p$ $(1 < p \leq 2)$ and replace condition (iii) on the real sequence $\{t_n\}$ by (iii)'': $\sum t_n^p < +\infty$. Then the same conclusion is obtained.

Corollary 4: In theorem 4, let $E = L_p$ $(2 \leq p < \infty)$ and replace (iii) by the condition (iii)": $\sum t_n^2 < +\infty$. Then the same conclusion is obtained.

Remark: If R(N) is bounded, then the boundedness of C is no longer required. Hence, in this case, we may take C = E. And in this case, the Hammerstein equation need not be homogeneous. That is, we will be solving (5) instead of (7).

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