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REPLICATION OF OPTIMAL CENTRAL COMPOSITE DESIGN: ROTATABILITY AND ORTHOGONALITY DESIGN RESTRICTIONS

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ABSTRACT

Replication is the repetition of the treatment under investigation to different experimental unit. Replication is essential for obtaining a valid estimate of the experimental error and to a greater extent, increasing the precision of estimating the pairwise difference among the treatment effect. It is on this note that when full replication (replication of cube and star point) is made, there exists need for optimal replication for the purpose of minimizing biasedness. Some variations of experimental points of central composite designs in the presence of complete replication are compared under rotatable and orthogonal design restrictions. The optimal choice of the points replicated is obtained using the A-, D- and E-optimality criteria. Comparisons of the variations are given with efficiency and the results suggest that replicated cubes plus replicated star points is better than partial replication of cube and star points under the design restrictions of rotatability and orthogonality.

1.0 INTRODUCTION

Some designs in response surface methodology such as first and second-order designs play vital role in modeling response function. Factorial designs are the most classical designs and they assume the regression model used as an approximating model to the true unknown response function, which is known as a full polynomial function, found in [1]. Sometimes, there are not enough resources to run a full factorial design, instead, one can run a fraction of the total number of treatments. Assuming we have, 2^{k-p} design that means that k factors each with 2-level, but run only 2^{k-p} treatments (as opposed to 2^k). 2^{4-1} Design is that design with 4 factors but run only $2^3 = 8$ treatments (instead of 16). That is $\frac{8}{16} = \frac{1}{2}$. That is why it is known as "$\frac{1}{2}$ replicate" or "half replicate". However, it is not all factor effects that can be estimated.

Obviously, factors are allowed with one another. Put differently, factors are confounded and one cannot estimate their effects separately. For example, if we consider factors A and D that forms aliased, when we

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estimate the effect for A, we are actually estimating the effect of A and D together. It is only further experimentation that can separate both A and D. main effect and low order interactions are of most interest and are usually more significant than high order interaction terms which turns zero as they increase according to investigation. This shows that by aliasing main effects with high order interactions, we can obtain a fairly accurate estimate of the main effects.

Many research works have been carried out on the replication of cube and star points of a central composite design partially. This work will seek to determine the efficiency and relative efficiency of cube and star replication in the presence of partial cube and star replication under the design restriction of orthogonality using A-, D-, E-, E_A and D_A -optimality criteria.

In many occasions, researches in central composite design have been carried out to solve one problem or the other. Determining the optimal (best) design using D-efficiency criterion has not been sufficiently explored and removing error through relative efficiency has not also been adequately researched on. Hence, this paper seeks to determine the optimal (best) design base on the efficiency using the D-efficiency criterion and also the reduction in the experimental error using relative efficiency of a complete replication of cube and star points of a central composite design under the design restrictions of rotatability and orthogonality.

In addition, this paper focuses on orthogonal central composite designs when the factors are two, three and four $\left(k=2, k=3, k=4\right)$ using replicated cube plus replicated star point to compare with ;(a) replicated cube plus one star point, (b) one cube plus replicated star point). The efficiencies and relative efficiencies for the above replications are also taken into consideration.

2. Literature Review

[2] studied the efficiencies of various second-order response surface designs, which includes second-order rotatable design, second-order slope rotatable designs, second-order rotatable design with an equi-spaces does, and second-order slope rotatable designs with an equi-space doses using symmetrical unequal block arrangements with two unequal block sizes for the estimation of responses and slope at different points (centre, axial and corner points) on second-order response surface designs. This was done as a result of the fact that some second-order rotatable designs and second-order slope rotatable designs constructed by using a symmetrical unequal block arrangement with two unequal block sizes have fewer design points compared to those constructed by other methods. A second-order response surface design considered here is given by

$$Y_u = b_0 + \sum_{i=1}^{k} b_i x_{iu} + \sum_{i=1}^{k} b_{ii} x_i^2 + \sum\sum_{i<j}^{v} b_{ij} x_{iu} x_{ju} + \varepsilon_u \qquad 2.1$$

where x_{iu} is the level of the ith factor in the uth run of the experiment, and ε_u 's are uncorrelated random error with mean zero and variance σ^2. b_0, b_i, b_{ii} and b_{ij} are the parameters of the model and Y_u is the response observed at the uth design point. The following conditions were imposed on the design points to simplify the solution of the normal equations:

$$\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \qquad 2.2$$

if any α_i is odd, for $\sum \alpha_i < 4$, then the following holds:

(i) $\sum_{u=1}^{N} x_{iu}^2 = \text{constant} = N\lambda_2$

(ii) $\sum_{u=1}^{N} x_{iu}^4 = \text{constant} = cN\lambda_4, \forall i$

(iii) $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4, \forall i \neq j$, where c, λ_2 and λ_4 are constants.

In their work, a conclusion was drawn that there is no uniform best design for the estimation of responses and slope at centre, axial and cube points on the second-order response surface for different values of v and p. [3] proposed an extended version of central composite design and named it central composite design 2. Conditions for orthogonality, rotatability and slope-rotatability for the central composite design 2 that have orthogonaility and rotatability property, orthogonality and slope-rotatability over axial directions and both rotatability and uniform precision. In this work, it was established as a known fact that the condition by which a central composite design is orthogonal is that

$$\alpha\left\{\frac{\sqrt{F\left(F+2k+n_0\right)}-F}{2}\right\}^{\frac{1}{2}} \qquad 2.3$$

where F is the factorial part of the design given by 2^k, 2k is the axial point and n_0 is the number of center point. Also, the condition for rotatability is that $\alpha = F^{\frac{1}{4}}$.

In slope rotatability, it was established that

$$\left(F+n_0\right)\alpha^8-4kF\alpha^6-F\left\{M\left(4-k\right)+kF-8\left(k-1\right)\right\}\alpha^4+8\left(k-1\right)F^2\alpha^2-2\left(k-1\right)F^2\left(M-F\right)=0 \qquad 2.4$$

where M is the total number of experimental run given by $\left(F+2k+n_0\right)$, F is the factorial part given by 2^k, where k is the number of factors and n_0 is the number of center point, depends on whether it is rotatability or orthogonality restriction. In [4] studied about a measure of rotatability for second order response surface designs. This measure of rotatability enabled them to assess the degree of rotatability for a given response surface design. 3^k factorial design and central composite design were used. The model used is given by

$$Y=\beta_0+\sum_{i=1}^{k}\beta_i x_i+\sum_{i=1}^{k}\beta_{ii}x_i^2+\sum_{i<j}^{k}\beta_{ij}x_i x_j+\varepsilon_i \qquad 2.5$$

[1] studied restricted second-order designs on one and two concentric balls. The results show that within the range of values of the ratio of the radii $\frac{g_1}{g_2}$, the ratio is given in the equation

$$\frac{g_1}{g_2}=\alpha \geq \left\{\frac{2^k r_1\left[N^{\frac{1}{2}}-\left(2^k r_1\right)^{\frac{1}{2}}\right]^2}{4r_2^2}\right\}^{\frac{1}{4}} \qquad 2.6$$

where g_1 is the radius of one concentric ball and g_2 is the radius of the other concentric ball. The orthogonal restriction was seen to be better (in the sense of smaller variance) than the unrestricted. Also, for some range, $\frac{g_1}{g_2}$ the orthogonal-rotatable constraint gives the best reduction in variance of linear function. This shows that in choosing restrictions on a quadratic design, where adequacy of the design is judged by the information matrix, the orthogonal-rotatable design is better than the rotatable, and orthogonal of the unconstrained. [5] studied two variations of N-point central composite designs that are either orthogonally or rotatably restricted. In the study, calculations were made based on the concept of Schur's ordering of designs or D-optimality criterion. The results show that replicated cube plus one star point variation is better that one cube plus replicated star point variation.

[6] studied the comparison of different central composites designs. Computations were carried out in terms of eigenvalues for the comparison of the class of central composite designs on the basis of E-optimality. It was found that the increase in the variable for central composite design when a combination of three observations is missing. Due to missing a combination of three observations, the increase in variance of the estimates is less as the number of design points is increased regardless of whether the missing combination consists of factorial,

axial or centre points. The missing of an axial point may create more problem than the missing of a factorial point measuring the variance for $k \geq 4$. If in experiment, a most informative combination of observation is missing the variance will then be more compared to a situation when a least informative combination of observation is missing. The most informative combination of missing observations increases the variance maximum as compared to a missing information combination of missing observations as in [7].

[8] studied the behavior of D-Optimal exact designs under changing regression polynomials. In the polynomials, some were with or without intercept. Also, some were with or without interactive terms were defined on design regions which were supported by the points of the circumscribed central composite designs. It was realized that the best N-point exact design for the intercept model (model I) is the same as the best N-point D-optimal design for the no-intercept model (model 3). Also, the best N-point D-optimal design for the intercept model(model 2) is the same as the best N-point D-optimal design for the no-intercept model(model 4) as measured by the determinant values, D-efficiencies, G-efficiencies and condition numbers. The equivalence of D-optimality and G-optimality criteria was established for the 4-point design under model 1, for the 2-point and 4-point designs. Under model 2, for the 4-point design under model 3 and for the 4-point design under model 4. This was further illustrated in [9] .

[10] studied a hill-climbing combinatorial algorithm for constructing N-point D-optimal exact designs. In the study, their interest was to study the difficulties encountered in the use of variance exchange algorithm in the construction of D-optimal exact design, which include cycling, slow convergence and failure to converge to the desire optimum. This was not experienced by this method. This method converges rapidly and absolutely to the desired N-point D-optimal design and is effective for determining optimal design in block experiment as well as in non-block experiments for finite or infinite number of support points in the space of trial.

[11] studied the imposition of D-optimality criterion on the design regions of the central composite designs. The investigation was done using second-order response surface model. It was realized that for the six parameter, second order polynomial model used, the D-optimal design defined over the rotatable (circumscribed) central composite design region has better determinant values than those obtained over the face-centered central composite design region and the inscribed central composite designs defined over the rotatable central composite design region gives a better parameter estimates as the variance and covariances of the parameters are minimised.

[1] studied the effects of changing design size, axial distances and increased center points for equiradial design with variation in model parameters. In their work, optimality criteria such as A-, D-, E-, G- and T- were put to use for full and reduces bivariate quadratic models alongside their efficiency. The full model was made to reduce by omitting the interaction term and the two models were compared by the use of the optimality criteria where center points lies between 1 and 5 inclusive. The results show that the relationship between D- and G-optimality criteria suggest that larger value of D-optimal design has smaller value of G-optimal design which in turn implies larger value of A-optimal and E-optimal designs. Also, a-optimality maintain a steady flow which is to say, constantly decreasing as center point increases, hence A-optimality criterion serves as the best criterion among the once studied for a reduced quadratic model. The D-optimality of equiradial designs increases for increasing axial distances for radial points n=5 and 1 center point for a reduced model which is also true for full model. It was also observed that D-optimality criterion of equiradial design for axial distance of 1.414 show superiority over equiradial design of 10 both for full and reduced model and it is true for all the radial and center points studied. The D-optimality of equiradial design is better for reduced model than for full model for all axial distance and center points studied. The implication is that equiradial design minimizes the variance of the parameters estimates for reduced model than for full model, [12].

[13] studied an alternative second-order N-point spherical response surface methodology designs and their efficiencies. In their work, the D-efficiency of the equiradial designs were evaluated with respect to the spherical central composite designs. Also, D-efficiencies of the equiradial designs were evaluated with respect to the D-optimal exact designs defined on the design region of the circumscribed central composite design, the inscribed central composite design and the face-centered central composite design. The D-efficiency values reveal that the alternative second-order N-point spherical central composite designs are better than the inscribed central composite design though inferior to the circumscribed central composite design with efficiency values less than 50% in all cases studied. Also, D-efficiency values reveal that the alternative second-order N-point spherical equiradial designs are better than the N-point D-optimal exact designs defined on the design region supported by the design points of the inscribed central composite designs. It was however, realized that the N-point spherical equiradial designs are inferior to the N-point D-optimal exact designs defined on the design region supported by the design points of the circumscribed central composite design and those of the face-centered central composite design with worse cases with respect to the design region of the circumscribed central composite design.

3. Methodology

The central composite design which is also known as the Box Wilson design was first developed by [14] and it appears to be the most popular class of second order model design which is given as

$$Y_{ij} = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i<j}^{k} \beta_{ii} x_i^2 + \sum_{i<j}^{k} \sum \beta_{ij} x_i x_j + \varepsilon_{ij} \qquad 3.1$$

where Y_{ij} is the measured response, $x_i, i = 1, 2, ...k$ represent the input variable, $\beta_0, \beta_i, \beta_{ii}$ and β_{ij} are the unknown parameters and ε_{ij} is the random error with mean as zero and variance of σ^2. A central composite design is made up of three different set of experimental runs but we will concentrate on two;

a. Rotatable Central Composite Design

[14] introduces the concept of rotatability. This concept is so essential while dealing with a second order design. A rotatable central composite design has a stable distribution as

$$\frac{N \operatorname{var}\left[\hat{y}(x)\right]}{\sigma^2} \qquad 3.2$$

where N is the number of observations made in accordance with the experimental design, $\operatorname{var}\left[\hat{y}(x)\right]$ is the variance of the estimated response at the point $x = (x_1, x_2, ...x_n)$ and $\hat{y}(x)$ is the estimated response at the point $x = (x_1, x_2, ...x_n)$.

Therefore, a design is said to be rotatable if the prediction variance is constant at all point ay x, which are the same distant from the centre of the design. One of the advantages of this property is that under any rotation of the coordinate axes, the prediction variance remains the same. Obviously, the concept of rotatability could be affected by moment and this moment is that of order four, [15]; [16]. Hence, [14] proposed that the necessary and sufficient conditions for a design to be rotatable are that all odd moments through order four should be considered as zero and also

$$\frac{\frac{1}{N}\sum_{u=1}^{N} x_u^4}{\frac{1}{N}\sum_{i=1}^{N} x_{iu}^4 x_{ju}^2} = \frac{f + 2\alpha^4}{f} = 3, (i \neq j) \qquad 3.3$$

Hence, a rotatable central composite design could be said to depend on the number of factorial points. A design then is said to be rotatable if it satisfies the condition that

$$\frac{n_f + 2\alpha^4}{n_f} = 3 \qquad 3.4$$

and

$$\alpha = \left(n_f\right)^{\frac{1}{4}} = \sqrt[4]{n_f} \qquad 3.5$$

where n_f is the number of factorial points. [17]. This also implies that for a rotatable central composite design, the value of α does not depend on the number of centre points but if each axial point is observed n_s times, then the requirement for rotatability is that

$$\alpha = \left(\frac{n_f}{n_s}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{n_f}{n_s}} \qquad 3.6$$

b. **Orthogonal Central Composite Design**

An orthogonal central composite design is that design in which the corresponding information matrix $(X'X)$ is diagonal. It is worthy of note that for a second order design, the diagonal of the information matrix $(X'X)$ seems impossible to obtain, but this can be tracked if we consider the model with the pure quadratic terms connected for their means. That is, $y_{ij} = \beta_0' + \sum_{i=1}^{k}\beta_i x_{iu} + \sum \beta_{ii}\left(x_{iu}^2 - \bar{x}_i^2\right) + \sum_{i<j}^{k}\sum_{i\neq j}^{k}\beta_{ij}x_{iu}x_j + \varepsilon_{ij}$ 3.7

$u = 1, 2, 3, \ldots N$, where in each case, $\beta_0' = \beta_0 + \sum_{i=1}^{k}\beta_{ii}\bar{x}_i^2$ and $\bar{x}_i^2 = \sum_{i=1}^{k}\left(\frac{x_i^2}{N}\right)$. Suppose b_0, b_i, b_{1i}, b_{1j} represent the least square estimators of the parameters $\beta_0, \beta_i, \beta_{1i}, \beta_{1j}$ respectively. In the central composite, all the covariance between the estimated regression coefficients except that of $\left(b_{ii}, b_{ij}\right)$, that is $\operatorname{cov}\left(b_{ii}, b_{ij}\right)$ becomes zero.

Assuming that the inverse of the information matrix forms a diagonal matrix $(X'X)^{-1}$, then also, $\operatorname{cov}\left(b_{ii}, b_{ij}\right)$ will be zero, [18]. Hence, a central composite design is said to possess a property of orthogonality when

$$\alpha = \left\{\frac{\sqrt{F\left(F + 2k + n_0\right)} - F}{2}\right\}^{\frac{1}{2}} \qquad 3.8$$

3.1 EFFICIENCY OF DESIGNS

In this paper, three designs (design one as complete replication of cube and star point, design two as partial cube replication and design three as partial star replication) were considered. For design one to be better (efficient) as compared with design two and three, it must happen that

$$\left|\left(X_1^1 X_1\right)^{-1}\right| < \left|\left(X_2^1 X_2\right)^{-1}\right| < \left|\left(X_3^1 X_3\right)^{-1}\right| \qquad 3.9$$

When there are two factors, we have 3-times cube and star replication, the result shows thus

$$\left|\left(X_1^1 X_1\right)^{-1}\right| = 4.2923e - 29, \left|\left(X_2^1 X_2\right)^{-1}\right| = 9.7569e - 8, \left|\left(X_3^1 X_3\right)^{-1}\right| = 1.2716e - 6 \cdot$$

It is obvious that

$$\left|\left(X_1^1 X_1\right)^{-1}\right| < \left|\left(X_2^1 X_2\right)^{-1}\right| < \left|\left(X_3^1 X_3\right)^{-1}\right|, \Rightarrow 4.2923e - 29 < 9.7569e - 8 < 1.2716e - 6.$$

This makes the designs that are replicated both at cube and star point to be better than the one with partial replication according to D-criterion. The summary of others are tabulated in Table 2. The design is not efficient when the replication is two and at two factors only. This calls for higher replication.

Efficiency of two Designs:

According to D-optimality, a design is said to be efficient it is has the least of the determinant of the inverse of the information matrix $(X'X)$. That is if

$$\left|\left(X_1^1X_1\right)^{-1}\right| < \left|\left(X_2^1X_2\right)^{-1}\right| < \left|\left(X_3^1X_3\right)^{-1}\right| < \ldots < \left|\left(X_n^1X_n\right)^{-1}\right| \qquad 3.10$$

it means that design one is efficient [19].

Relative Efficiency of Designs

Relative efficiency is used to compare different experimental designs with respect to the reduction of experimental error. In order words, relative efficiency is used to determine the extent of the reduction in the experimental error of one design relative to another [19] It is used to find out the degree of reduction in the experimental error of design one relative to design two and three as the case may be. This is achieved by using the condition

$$E_D = \left[\frac{\left|\left(X_1^1X_1\right)^{-1}\right|}{\left|\left(X_2^1X_2\right)^{-1}\right|}\right]^{\frac{1}{p}} \qquad 3.11$$

where p is the number of parameters, i is design one and j is designs 2 and 3 respectively.

Table1: Factors, Replication, Optimality Values and Optimality Criteria

Factors	No.of rep.	Values of optimality	Optimality Criteria
2	2-times	Max{143.62,96.32,32.57}=143.62	A-Optimality
	3-times	Max{270.56,138.86,109,}=270.56	
	4-times	Max{433,181,137}=433	
3	2-times	Max{413,287,222.93}=413	
	3-times	Max{779.17,420.40,293.91}=779.17	
	4-times	Max{1248.8,552.93,364.89}=1248.8	
4	2-times	Max{1099.5,790.24,577}=1099.5	
	3-times	Max{2063.3,1168.4,745}=2063.3	
	4-times	Max{3297,1545,913}=3297	
2	2-times	Max{6.7045e6,1.0225e6,163840}=6.7045e6	D--Optimality
	3-times	Max{2.3297e8,1.0249e7,786432}=2.3297e8	
	4-times	Max{3.1709e9,56623104,2048000}=3.1709e9	
3	2-times	Max{1.7856e15,1.0669e14,4.4394e12}=1.7856e15	
	3-times	Max{4.0519e17,4.5752e15,3.5703e13}=4.0517e17	
	4-times	Max{1.9863e19,6.8039e16,1.7501e14}=1.9863e19	
4	2-times	Max{2.5159e26,7.7961e24,1.9480e22}=2.5159e26	
	3-times	Max{6.589e29,1.6547e27,2.4927e23}=6.5893e29	
	4-times	Max{1.8444e32,1.6547e29,1.7572e24}=1.8444e32	
2	2-times	Max{1.0993,0.9723,3.0873}=3.0873}	E-Optimality
	3-times	Max{2.1107,1.5656,0.6620}=2.1107}	
	4-times	Max{3.5474,2.3223,0.6629}=3.5475	
3	2-times	Max{16,16,8}=16	
	3-times	Max{24,24,8}=24	
	4-times	Max{32,32,8}=32	
4	2-times	Max{32,32,16}=32	
	3-times	Max{48,48,16}=48	

	4-times	Max{64,64,16}=64	
ORTHOGONALITY RESTRICTION			
2	2-times	Max{98.46,62.24,66.46}=98.46	A-Optimality
	3-times	Max{201.84,86.78,116.58}=201.84	
	4-times	Max{363.6,111.08,189.02}=363.6	
3	2-times	Max{298.34,191.75,196.59}=298.34	
	3-times	Max{629.14,272.8,361.11}=629.14	
	4-times	Max{1161.5,353.61,612.18}=1161.5	
4	2-times	Max{788.72,541.52,507.60}=788.72	
	3-times	Max{1631.3,783.32,934.76}=1631.3	
	4-times	Max{2980,1024.3,1617}	
2	2-times	Max{7.696e5,6.3206e4,9.3482e4}=7.696e5	D-Optimality
	3-times	Max{3.9026e7,2.8442e5,1.0732e6}=3.9026e7	
	4-times	Max{1.2429e9,8.2942e5,9.3590e6}=1.2429e9	
3	2-times	Max{1.7497e14,9.0677e11,1.8216e12}=1.7497e14	
	3-times	Max{1.1277e17,1.9039e13,1.2669e14}=1.1277e17	
	4-times	Max{1.3391e19,1.7563e14,3.4487e15}=1.3391e19	
4	2-times	Max{9.3315e24,7.3375e21,5.8024e21}=9.3315e24	
	3-times	Max{9.9853e28,8.3691e23,1.5000e24}=9.9853e28	
	4-times	Max{8.9140e31,2.4684e25,1.1597e26}=8.9140e31	
2	2-times	Max{0.6682,1.0327,0.6816}=1.0327	E-Optimality
	3-times	Max{1.0841,1.0463,0.6891 }=1.0841	
	4-times	Max{2.5888,1.0414,0.8853}=2.5888	
3	2-times	Max{16,5.07,8}=16	
	3-times	Max{ 24,5.32,8 }=24	
	4-times	Max{ 32,5.54,8}=32	
4	2-times	Max{ 32,8.91,16}=32	
	3-times	Max{48,9.47,16}=48	
	4-times	Max{64,9.47,16}=64	

Obviously, for a design to be a-optimal, it means that it must satisfy the criterion $A = \max tr(X'X)$. This means finding the traces of the information matrix for each design and check out of these traces, which is maximum. From table 1, we observed the traces to be {143.62, 96.32, 57} $= tr(X'X)$. Then $\max tr(X'X) = 143.62$. This is a design with cube and star replication. This shows clearly that design one is a-optimal. In the same manner, Table 1 also shows the optimal D-values under the design restriction of rotatability. The criterion for a design to be D-optimal is that out of the determinants of the information matrices, the design that is D-optimal should have maximum determinant. From Table 1, we observed that $\det(X'X) = \{6.7045e6, 1.0225e6, 163840\}$, then $\max\det(X'X) = 6.7045e6$ which occurs when the design is replicated both at cube and star points. Hence, the design is D-optimal. Also, Table 1 shows the optimal E-values under the design restriction of rotatability. The criterion for a design to be E-optimal is that out of the minimum eigen values, the design must have maximum eigen value. From Table 1, we observed that $\max\left[\min(e_i)\right] = \max[1.0993, 0.9723, 3.0873] = 3.0873$. This is when replication is two and when there are two factors. The design fails to be E-optimal. When the replication is increased, the design remains E-optimal like when the replication is three, we have

$\max\left[\min\left(e_i\right)\right] = \max\left[2.1107, 1.5656, 0.6620\right] = 2.1107$ which occurs at three time replication. This is one of the reasons why we embark on replication to know its important in design. Hence, this result calls for higher replication of cube and star points for design to be E-optimal.

2-times replication of cube and star points means that the cube is replicated 2-times and the star point is also replicated 2-times. 2-times cube plus one star point means it is only the cube that is replicated 2-times. One cube plus one star point replication means that it is only star point that is replicated 2-times. This follows 3-times and 4-times at 2, 3, and 4 factors respectively.

2-times replication of cube and star point under E-optimality criterion fails to be E-optimal since if did not satisfy the criterion of $\max\left[\min\left(e_i\right)\right]$. The minimum eigenvalue for replication of cube and star point is 1.0993, the minimum eigenvalue for partial replication of cube point is 0.9723 and the minimum eigenvalue for partial replication of star point is 3.0873. Then, taking the maximum of the minimum as arranged in this project starting with replication of cube and star point, followed by partial cube replication and finally partial star point, we have $\max\left[1.0993, 0.9723, 3.0873\right] = 3.0873$. This occurs at partial replication of star point. This is why we conclude that the design fail to be E-optimal at 2-times replication of cube and star points when there are two factors and at the design restriction of rotatability. Hence, in this case it calls for higher replication.

For the relative efficiency, design 2(2times cube plus one star point replication) is 73% as efficient as design 1(cube and star replication) and also, design 3(one cube plus 2-times replication of star point) is 54% as efficient as design 1(cube and stat point replication). In the same manner, design 2 is 59% as efficient as design 1 and design 3 is 39% as efficient as design 1 when replication is done three times. In the same way, design 2 is51% as efficient as design 1 and design 3 is 29% as efficient as design 1. Generally, we observed a trend where cube and star point replication is better than a design with partial cube and partial star points replication.

We establish their relative efficiencies as follows:

Table2: Factors, Replication, Efficiency values and Comments

FACTORS	NO. OF REP	EFFICIENCY COMPARED $\left\|\left(X_1^1 X_1\right)^{-1}\right\| < \left\|\left(X_2^1 X_2\right)^{-1}\right\| < \left\|\left(X_3^1 X_3\right)^{-1}\right\|$	DESIGN RESTRICTION	COMMENT
2	2-times	1.4913e-7>9.7814e-7<6.1035e-6	**ROTATABILITY**	NOT EFFICIENT
	3-times	4.2923e-29<9.7569e-8<1.2716e-6		EFFICIENT
	4-times	3.1537e-10<1.7661e-8<4.8828e-7		EFFICIENT
3	2-times	5.6003e-16<9.3732e-15<2.2526e-13		EFFICIENT
	3-times	2.4680e-18<2.1857e-16<2.8009e-14		EFFICIENT
	4-times	5.0345e-20<1.4697e-17<5.71339e-15		EFFICIENT
4	2-times	3.9747e-27<1.2827e-27<5.1335e-23		EFFICIENT
	3-times	1.5176e-30<3.8444e-28<4.0116e-24		EFFICIENT
	4-times	5.4217e-33<6.0433e-30<5.6910e-25		EFFICIENT
2	2-times	1.2994e-5<1.5821e-5<1.0697e-5	**ORTHOGONALITY**	EFFICIENT
	3-times	2.5624e-8<3.4714e-6<9.3180e-7		EFFICIENT
	4-times	8.0456e-10<1.2057e-7<1.0735e-7		EFFICIENT
3	2-times	5.7155e-15<1.1028e-12<5.2109e-13		EFFICIENT
	3-times	8.8673e-18<5.2524e-14<7.8988e-15		EFFICIENT
	4-times	7.4680e-20<5.6937e-15<2.8997e-16		EFFICIENT
4	2-times	1.0716e-25<1.3629e-22<1.7234e-22		EFFICIENT
	3-times	1.0015e-29<1.1949e-24<6.6666e-25		EFFICIENT
	4-times	1.1218e-32<4.0511e-26<8.6227e-25		EFFICIENT

Table 3: Factors, Variations in Replication and Relative Efficiency Values

ROTATABILITY RESTRICTION

Factors	Variation of Replication	Relative efficiency
2	2-times cube and stat replication with 2-times cube plus one star point replication	0.7309
	2-times cube and stat replication with one cube plus 2-times star point replication	0.5386
	3-times cube and stat replication with 3-times cube plus one star point	0.5941
	3-times cube and stat replication with one cube plus 3-times star point replication	0.3874
	4-times cube and stat replication with 4-times cube plus one star point	0.5113
	4-times cube and stat replication with one cube plus 4-times star point replication	0.2940
3	2-times cube and stat replication with 2-times cube plus one star point replication	0.7545
	2-times cube and stat replication with one cube plus 2-times star point replication	0.5489
	3-times cube and stat replication with 3-times cube plus one star point	0.6386
	3-times cube and stat replication with one cube plus 3-times star point replication	0.3931
	4-times cube and stat replication with 4-times cube plus one star point	0.5669
	4-times cube and stat replication with one cube plus 4-times star point replication	0.3123
4	2-times cube and stat replication with 2-times cube plus one star point replication	0.7933
	2-times cube and stat replication with one cube plus 2-times star point replication	0.5320
	3-times cube and stat replication with 3-times cube plus one star point	0.6914
	3-times cube and stat replication with one cube plus 3-times star point replication	0.3731
	4-times cube and stat replication with 4-times cube plus one star point	0.6264
	4-times cube and stat replication with one cube plus 4-times star point replication	0.2919
2	2-times cube and stat replication with 2-times cube plus one star point replication	0.6593
	2-times cube and stat replication with one cube plus 2-times star point replication	0.3266
	3-times cube and stat replication with 3-times cube plus one star point	0.4413
	3-times cube and stat replication with one cube plus 3-times star point replication	0.5494
	4-times cube and stat replication with 4-times cube plus one star point	0.2956
	4-times cube and stat replication with one cube plus 4-times star point replication	0.4424
	2-times cube and stat replication with 2-times cube plus one star point replication	0.5908
	2-times cube and stat replication with one cube plus 2-times star point replication	0.6752
	3-times cube and stat replication with 3-times cube plus one star point	0.4195
	3-times cube and stat replication with one cube plus 3-times star point	0.5070

3	replication	
	4-times cube and stat replication with 4-times cube plus one star point	0.3249
	4-times cube and stat replication with one cube plus 4-times star point replication	0.4376
4	2-times cube and stat replication with 2-times cube plus one star point replication	0.6209
	2-times cube and stat replication with one cube plus 2-times star point replication	0.6113
	3-times cube and stat replication with 3-times cube plus one star point	0.4587
	3-times cube and stat replication with one cube plus 3-times star point replication	0.4769
	4-times cube and stat replication with 4-times cube plus one star point	0.3654
	4-times cube and stat replication with one cube plus 4-times star point replication	0.4052

4. CONCLUSION

In this project, calculations have been made to determine whether a design that is replicated both at cube and star points is A-, D-, and E-optimal. In other words, using the criterion for A-optimality $A = \max tr\left(X'X\right)$, we have seen that of the traces, the maximum is at design 1(cube and star replication). In the same manner, under the criterion of D-optimality $A = \max \det\left(X'X\right)$, we noticed that the maximum determinant out of the three designs comes from design 1(cube and star point replication) which makes it D-optimal. Finally, for E-criterion, the maximum of the minimum eigenvalues comes from design 1, except 2-times replication which makes design one E-optimal. Hence, our first objective achieved.

We also calculated the efficiency of the designs (cube and star point replication, partial replication of cube and star point) and the results show clearly that it is at 2-times replication that the design fails to be efficient, but at 3-times and 4-times replication, the design that is replicated at cube and star points remains efficient at both design restrictions of rotatability and orthogonality. This is tabulated in Table 2. For design 1 to be efficient, $\left|\left(X_1^1 X_1\right)^{-1}\right| < \left|\left(X_2^1 X_2\right)^{-1}\right| < \left|\left(X_3^1 X_3\right)^{-1}\right|$. This means that design one is efficient since it has the least value.

Relative efficiency was also calculated to compare experimental designs with respect to the reduction in the experimental error. The results show that under the design restriction of rotatability, there is a minimal error when the replication is done at both cube and star points relative to partial cube replication than when replication is done at both cube and star points relative to partial star replication under the design restriction of orthogonality. Hence, we should prefer cube replication to star replication if at all we wish to embark on partial replication.

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