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STATISTICAL PROPERTIES AND APPLICATION OF THE EXPONENTIAL DAGUM LOG-LOGISTIC DISTRIBUTION

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ABSTRACT

In this paper, we propose a new distribution from the T-R{Y} family of distributions called the Exponential Dagum Log-logistic (EXDAL) distribution. The statistical properties of the proposed distribution are carefully discussed. The maximum likelihood method of parameter estimation is employed for the estimates of the parameters of the proposed distribution. The flexibility of the proposed distribution is demonstrated using a real life dataset. Result from the application of the proposed distribution reveals that it performs better than the Dagum and Log Logistics distribution in fitting real life dataset.

1. Introduction

Several probability distributions in literature have been used to fit and model lifetime data for the purpose of accurate forecasting and planning. Statistical models are useful in modeling and resolving real life problems. This has led to the development of new and adaptive model by various researchers in recent times. New statistical model has been developed either as generalized models or combination of existing models, with intention to create a more robust and adaptive model for real life applications. More so, the development of new flexible and highly adaptive probability distributions alongside the means of their estimation has been on the background of this development, [1]. Recent developments focus on new techniques by compounding of distributions and adding parameters to existing distributions thereby building classes of more flexible distributions.

The Lambda distribution was proposed by [2] using a method based on quantile function, which was generalized by [3], [4] and called the Generalized Lambda Distributions (GLD). This family of distributions is defined in terms of percentile function. The Dagum distribution was proposed by [5] to fit empirical income and wealth data that could accommodate heavy tails in income and wealth distributions. For an extensive review on the genesis and empirical applications of the Dagum distribution, see [6]. The four parameter distribution called the Dagum-Poisson

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(DP) distribution was introduced by [7]. This distribution is obtained by compounding the Dagum and Poisson distributions.

The Mc-Dagum distribution was proposed and studied by [8]. The Transmuted Dagum and the Log-Dagum distributions were proposed respectively by [9] and [10]. The Beta-Dagum distribution was introduced and its properties studied by [11]. The Exponentiated Generalized Exponential Dagum Distribution was introduced by [12]. The history of exponential density function can be traced back to the work from various researchers such as [13]. By compounding the exponential distribution and the Poisson distribution, the Exponential-Poisson (EP) distribution was developed by [14]. A two-parameter Exponential-Geometric (EG) distribution developed by compounding the exponential and geometric distributions was pioneered by [15]. The Exponential-Logarithmic (EL) distribution was obtained by [16] compounding the exponential distribution and the logarithmic distribution.

Nadarajah (2006) defined the exponentiatedGumbel distribution. Mansoor et al. (2018) introduced a three-parameter extension of the exponential distribution which contains sub-models to the exponential, logistic-The The Log-Logistic distribution is a very popular and widely used model in many areas such as reliability, survival analysis, Actuarial science, Economics, Engineering and Hydrology. The Extended Log-logistic distribution was proposed by [17] while the Marshal-Olkin Extended Log-logistic Distribution was proposed by [18]. The formulation of a flexible statistical distribution with interval (0,∞) is of great importance in statistical research. Numerous works in the literature have shown that the Dagum distribution provides better fit for dataset with support $(0, \infty)$. This article therefore proposes a new distribution called the Exponential-Dagum Log logistic distribution using the T-R{Y} framework. An application of the proposed distribution shows its flexibility in fitting real life data set.

2. Methodology

2.1 The T-R{Y} Family oF distribution

Let T, R, Y be random variables with their respective Cumulative Density Functions (CDF) given as

$F_T(x) = P(T \leq x), F_R(x) = P(R \leq x), F_Y(x) = P(Y \leq x)$ (1)

The quantile functions for the random variables T, R, Y are denoted respectively as:

$$Q_T(p), Q_R(p), Q_Y(p)$$

where p lies between zero and one.

The corresponding Probability Density Function (PDF) of T, R, Y are given respectively by $f_T(x), f_R(x) and f_Y(x)$

The "T-R{Y} family of distributions" was proposed by Alzaatreh et.al (2014) where the random variable X has a CDF defined as:

$$F(x) = \int_a^{Q_Y(F(x))} f_T(t)dt = F_T(Q_Y(F_R(x))) \qquad (2)$$

and$Q_Y(.)$ satisfies the conditions:

i. $Q_Y(F(x)) \in [a, b]$

ii, Q_Y is differentiable and monotonically non-decreasing

iii. $Q_Y(F(x)) \rightarrow a \text{ as } x \rightarrow -\infty \text{ and } Q_Y\big(F(x)\big) \rightarrow b \text{ as } x \rightarrow \infty$

The corresponding PDF is given as:

$f(x) = f_T(Q_Y(F_R(x))) \times Q'_Y(F_R(x)) \times f_R(x)$(3)

2.2 The Exponential Dagum Log-Logistic Distribution (EXDAL)

Let T be an Exponential random variable having CDF and PDF given respectively as:

$$F_T(x) = 1 - e^{-\theta x} \qquad (4)$$

$$f_T(x) = \theta e^{-\theta x}, \quad \theta > 0 \qquad (5)$$

whereθ is a rate or inverse scale parameter which is defined as the reciprocal of the scale parameter. The quantile function $Q_T(p)$ of the Exponential distribution by equating its CDF to p and solving for x. Hence, we have

$$Q_T(p) = \frac{-\log(1-p)}{\theta} \quad 0 < p < 1 \qquad (6)$$

Let R be the Dagum random variable whose CDF and PDF are defined respectively as:

$$F_R(x) = \left(1 + \left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta} \quad (7)$$

$$f_R(x) = \frac{dF_R(x)}{dx} = \frac{\beta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{-\alpha-1}\left(1 + \left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta-1} \quad (8)$$

$\beta, \alpha, \lambda, x > 0$,where$\beta, \alpha$ *are shape parameters and*λ *is a scale* parameter.
The quantile function $Q_R(p)$ of the Dagum distribution is obtained as:

$$Q_R(p) = \lambda\left[p^{-\frac{1}{\beta}} - 1\right]^{-\frac{1}{\alpha}} \quad (9)$$

Let Y be a Log logistic random variable whose CDF and PDF with are given respectively as

$$F_Y(x) = \frac{1}{1+\left(\frac{x}{\lambda}\right)^{-\beta}} = \left[1 + \left(\frac{x}{\lambda}\right)^{-\beta}\right]^{-1} \quad (10)$$

where$\beta, \lambda = 1$
Then,

$$F_Y(x) = [1 + x^{-1}]^{-1} = \left[\frac{x+1}{x}\right]^{-1} = \frac{x}{x+1} \quad (11)$$

and

$$f_Y(x) = \frac{dF_Y(x)}{dx} = \frac{1}{(1+x)^2} \quad (12)$$

We now obtain the quantile function of the Log logistic distribution $Q_Y(p)$ as:

$$Q_Y(p) = \frac{p}{1-p} \quad (13)$$

Thus, the CDF of the EXDAL distribution is given as:

$$F_X(x) = F_T\left(Q_Y\left(F_R(x)\right)\right) = F_T\left[\frac{F_R(x)}{1-F_R(x)}\right] \quad (14)$$

which gives:

$$F_X(x) = 1 - exp\left[-\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)\right] \quad (15)$$

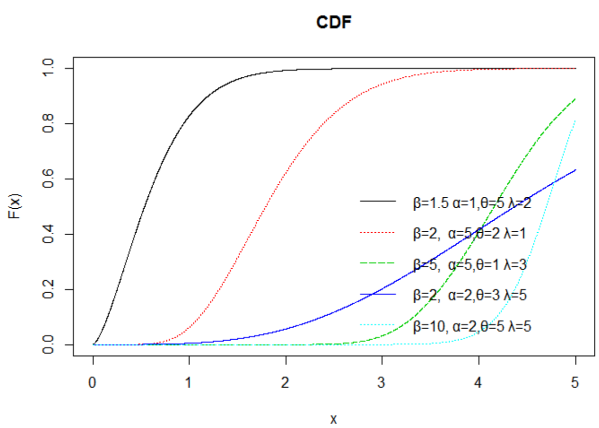


Figure 1: Plot of the Cumulative Density Function of the EXDAL Distribution for some selected parameter values

The corresponding PDF obtained by differentiating equation (15) w,r,t,x is given as:

$$f_x(x) = \frac{\frac{\beta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{-\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta-1}}{\left(1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right)^2} \quad \theta exp\left[-\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)\right] \quad (16)$$

$where \quad \beta, \alpha, > 0\ are\ shape\ parameters$
$\lambda\ > 0\ are\ scale\ parameter$
$\theta > 0\ \ is\ the$ rate parameter
$0 < x < 1$

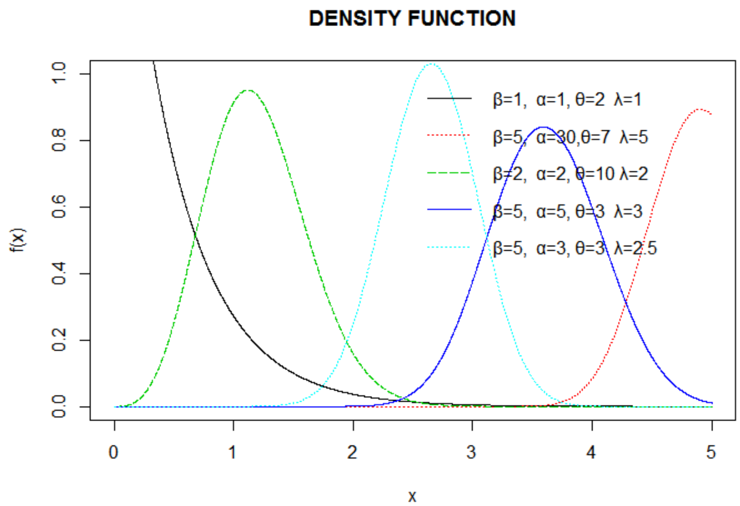


Figure 2: Plot of the PDF of the EXDAL Distribution for some selected parameter values.

2.3 Statistical Properties of the Exponential Dagum Log-Logistic Distribution

In this section, we carefully study and discuss some of the statistical properties of the EXDAL distribution.

2.3.1. Quantile Function of the EXDAL Distribution

To find the quantile function of the EXDAL distribution, the CDF obtained in (15) is equated to p and then solved for x to get:

$$p = 1 - exp\left[-\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)\right] \qquad (17)$$

$$\log(1-p) = -\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)$$

$$\theta\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta} = -\log(1-p) + \log(1-p)\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta} \qquad (18)$$

$$1+\left(\frac{x}{\lambda}\right)^{-\alpha} = \left(\frac{\log(1-p)}{\log(1-p)-\theta}\right)^{-\frac{1}{\beta}}$$

$$x = \lambda\left(\left(\frac{\log(1-p)}{\log(1-p)-\theta}\right)^{-\frac{1}{\beta}} - 1\right)^{-\frac{1}{\alpha}} \qquad (19)$$

So that

$$Q_x(p) = \lambda\left(\left(\frac{\log(1-p)}{\log(1-p)-\theta}\right)^{-\frac{1}{\beta}} - 1\right)^{-\frac{1}{\alpha}} \qquad (20)$$

Setting $p = 0.5$ gives the median as:

$$Q_x(0.5) = \lambda\left(\left(\frac{\log(1-0.5)}{\log(1-0.5)-\theta}\right)^{-\frac{1}{\beta}} - 1\right)^{-\frac{1}{\alpha}} \quad (21)$$

2.3.2. Survival Function of the EXDAL Distribution

The survival function of the EXDAL distribution is given as:

$$S(x) = 1 - F(x)$$

$$= 1 - \left[1 - exp\left[-\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)\right]\right] \quad (22)$$

which gives

$$S(x) = exp\left[-\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)\right] \quad (23)$$

$\alpha, \beta > 0$ are shape parameters while θ, λ are scale parameters

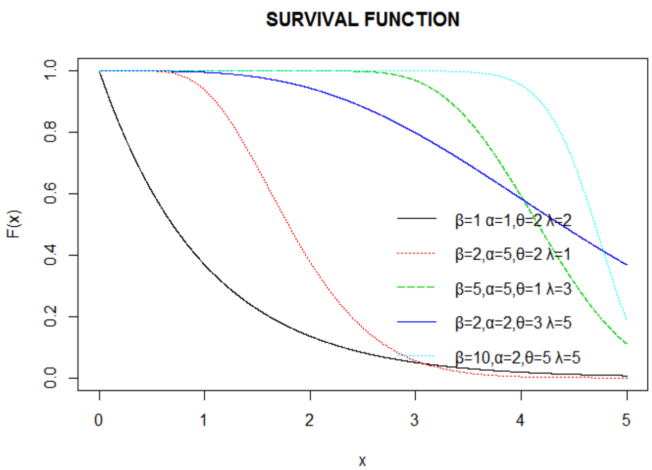


Figure 3: Survival Function of the EXDAL distribution for some selected parameter values.

2.3.3. Hazard Function of the EXDAL Distribution

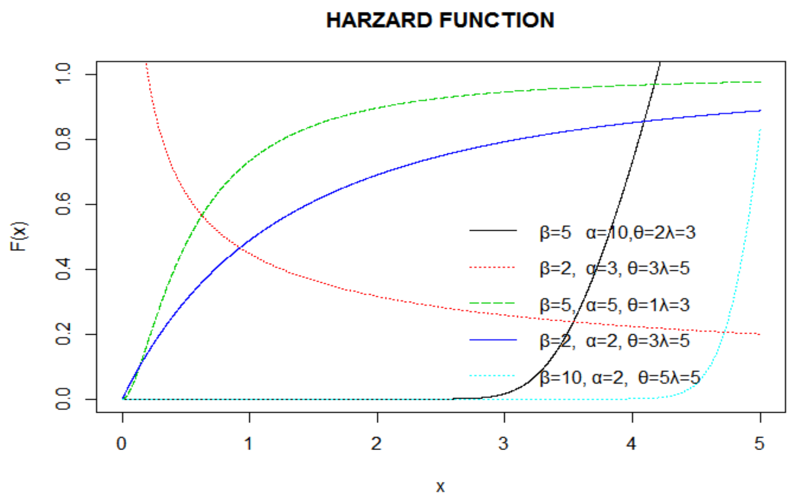
The hazard function of the EXDAL distribution is expressed as

$$h(x) = \frac{F(x)}{1 - F(x)}$$

$$= \frac{\dfrac{\frac{\beta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{-\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta-1}}{\left(1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right)^{2}}\theta exp\left[-\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)\right]}{exp\left[-\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)\right]} \quad (24)$$

Figure 4: Hazard Function of the EXDAL distribution for some parameter values

2.3.4. Moments of the EXDAL distribution



Before examining the moments of the EXDAL distribution, we first study the relationship between the Exponential and EXDAL distributions using the transformation technique. This is given in the theorem below:

Theorem 1:

If T is an exponential random variable, then the random variable $X = \lambda\left[\left(\frac{T}{1+T}\right)^{-\frac{1}{\beta}} - 1\right]^{-\frac{1}{\alpha}}$ is an Exponential-Dagum Log-logistic random variable having parameters α, β and λ

Proof

If T follows an exponential random variable, we can show that from

$T = Q_Y\left(F_R(X)\right)$

$$Q_Y\left(F_R(X)\right) = \frac{\left(1+\left(\frac{X}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{X}{\lambda}\right)^{-\alpha}\right)^{-\beta}} = \mathrm{T}$$

$$T = \left(1+\left(\frac{X}{\lambda}\right)^{-\alpha}\right)^{-\beta} + T\left(1+\left(\frac{X}{\lambda}\right)^{-\alpha}\right)^{-\beta}$$

$$X = \lambda\left[\left(\frac{T}{1+T}\right)^{-\frac{1}{\beta}} - 1\right]^{-\frac{1}{\alpha}} \qquad (25)$$

2.3.5. Moment of the EXDAL Distribution

The non-central moment of the EXDAL Distribution is given by

$$\mu_r = E\left[\left(\lambda\left[\left(\frac{T}{1+T}\right)^{-\frac{1}{\beta}} - 1\right]^{-\frac{1}{\alpha}}\right)^r\right] \qquad (26)$$

$$\text{Let } X = \lambda\left[-1+\left(\frac{T}{1+T}\right)^{-\frac{1}{\beta}}\right]^{-\frac{1}{\alpha}} \qquad (27)$$

Where $x = \left(\frac{T}{1+T}\right)^{-\frac{1}{\beta}}$ & $y = (-1)$,

Using the binomial formula written as

$$(x+y)^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}y^k \qquad (28)$$

We have that

$$X = \lambda \sum_{k=0}^{\infty} \binom{-\frac{1}{\alpha}}{k} \left[\left(\frac{T}{1+T} \right)^{-\frac{1}{\beta}} \right]^{-\frac{1}{\alpha}-k} (-1)^k$$

$$X = \lambda \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{\alpha}}{k} [1-(1+T)^{-1}]^{\frac{k+\frac{1}{\alpha}}{\beta}}$$

$$X = \lambda \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{\alpha}}{k} \sum_{m=0}^{\infty} (-1)^m \binom{\frac{k+\frac{1}{\alpha}}{\beta}}{m} [(1+T)^{-1}]^m$$

$$X = \lambda \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{k+m} \binom{-\frac{1}{\alpha}}{k} \binom{k+\frac{1}{\alpha}}{m} (1+T)^{-m} \qquad (29)$$

Hence

$$E(X) = kE[(1+T)^{-m}]$$

Using the binomial formula, we have that

$$(1+T)^{-m} = \sum_{p=0}^{\infty} \binom{-m}{p} T^p$$

so that

$$E(X^r) = \lambda \sum_k^{\infty} \sum_m^{\infty} \sum_p^{\infty} (-1)^{k+m} \binom{-\frac{1}{\alpha}}{k} \binom{k+\frac{1}{\alpha}}{m} \binom{-m}{p} E(T^{pr}) (30)$$

Let $pr = q$ and

$$A = \lambda \sum_k^{\infty} \sum_m^{\infty} \sum_p^{\infty} (-1)^{k+m} \binom{-\frac{1}{\alpha}}{k} \binom{k+\frac{1}{\alpha}}{m} \binom{-m}{p}$$

then

$$E(X^r) = AE(T^q)$$

Let $T \sim \exp(\lambda)$ so that $f_T(x) = \lambda e^{-\lambda t}$ so that

$$E(T^q) = \int_0^{\infty} t^q \lambda e^{-\lambda t} dt = \lambda \int_0^{\infty} t^q e^{-\lambda t} dt$$

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt$$

Let $\lambda t = v \Rightarrow t = \frac{v}{\lambda}$,

$$E(T^q) = \lambda \int_0^{\infty} \left(\frac{v}{\lambda}\right)^q e^{-v} \frac{dv}{\lambda} = \frac{1}{\lambda^q} \int_0^{\infty} v^q e^{-v} dv$$

$where \quad n-1 = q$

$$= \frac{1}{\lambda^q} \int_0^{\infty} v^q e^{-v} dv = \frac{1}{\lambda^q} \Gamma(q+1)$$

Hence,

$$E(X^r) = A \frac{1}{\lambda^q} \Gamma(q+1)$$

$$E(X)^r = \lambda^r \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{k+m} \binom{-\frac{1}{\alpha}}{k} \binom{\frac{k+\frac{1}{\alpha}}{m}}{} \binom{-m}{p} \frac{1}{\lambda^q} \Gamma(q+1) (31)$$

The mean of the EXDAL Distribution can be obtained by setting r =1 to get

$$\mu_1' = \lambda \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{k+m} \binom{-\frac{1}{\alpha}}{k} \left(\frac{k+\frac{1}{\alpha}}{m} \right) \binom{-m}{p} \Gamma(q+1)$$

where $r \neq q$ and $q = pr$

The variance of the EXDAL distribution is given as:

Variance (x) = $u_2' - (u_1')^2$

where

$$\mu_2' = \lambda^2 \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{k+m}\binom{-\frac{1}{\alpha}}{k}\binom{k+\frac{1}{\alpha}}{m}\binom{-m}{p}\Gamma(q+1)$$

2.3.6 Skewness and Kurtosis of the EXDAL Distribution

The skewness and kurtosis of the EXDAL distribution are given respectively as:

$$\boldsymbol{Skewness} = \frac{\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}{\left(\mu_2' - (\mu_1')^2\right)^{\frac{3}{2}}} \qquad (32)$$

$$\boldsymbol{kurtosis} = \frac{\mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4}{\left(\mu_2' - (\mu_1')^2\right)^2} \qquad (33)$$

2.3.7 Moment Generating Function of the EXDAL Distribution

Let X be a random variable having an EXDAL distribution. The moment generating function of X is given by:

$$M_x(t) = \lambda^q \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{k+m}\frac{t^q}{q!}\binom{-\frac{1}{\alpha}}{k}\binom{k+\frac{1}{\alpha}}{m}\binom{-m}{p}\Gamma(q+1) \quad (34)$$

2.3.8. Shannon Entropy of the EXDAL Distribution

If a random variable X follow the family of distribution with density function given as

$$\eta = E[-\log(f(X))] \quad (35)$$

$$f(x) = f_R(x)\frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))}$$

$$(Q_Y(F_R(x)) = \frac{F_R(x)}{1 - F_R(x)} = T$$

$$T = \frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}$$

$$\eta = E\left[-\log\left\{f_R(X)\frac{f_T(Q_Y(F_R(X)))}{f_Y(Q_Y(F_R(X)))}\right\}\right]$$

$$\eta = E\left[-\log\left\{f_R(x)\frac{f_T(T)}{f_Y(T)}\right\}\right]$$

Hence, the Shannon Entropy is:

$$\eta = E\left[-[log f_R(X) + log f_T(T) - log f_Y(T)]\right]$$

$$\eta = E[-log f_R(X)] + E[-log f_T(T)] + E[log f_Y(T)]$$

$$f_R(x) = \frac{\beta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{-\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta-1}$$

So that

$$-\log f_R(x) = -\log\beta - \log\alpha + \log\lambda + (\alpha-1)[\log(X) - \log(\lambda)] + (\beta-1)\log\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)$$

$$E(-\log f_R(X)) = -\log\beta - \log\alpha + \log\lambda + (\alpha-1)E(\log X) - (\alpha-1)\log(\lambda) + (\beta-1)E\log\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)$$

$$\eta = E(-\log f_R(X)) + E(\log f_Y(T)) - \log\beta - \log\alpha + \log\lambda + (\alpha-1)E(\log X) - (\alpha-1)\log(\lambda) + (\beta-1)E\log\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)$$

$$\eta = E(-\log f_T(T)) + E(\log f_Y(T)) - \log\beta - \log\alpha + \log\lambda + (\alpha-1)E(\log X) - (\alpha-1)\log(\lambda) + (\beta-1)E\log\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right) \quad (36)$$

where

$E(-\log f_T(T))) = \eta_T$

$E(\log f_Y(T)) = -\mu_T$

and by substitution we have

$$\eta = \eta_T - \mu_T - \log\beta - \log\alpha + \log\lambda + (\alpha-1)E(\log X) - (\alpha-1)\log(\lambda) + (\beta-1)E\log\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)$$

where

$$\eta_T \text{ is the Shannon entropy of the Exponential distribution defined by}$$
$$\eta_T = 1 - In\lambda$$

and

$$\mu_T = E(\log f_Y(T)) = E[\log\left(\frac{1}{(1+T)^2}\right)] = \log E(1+T)^{-2} = -2\log E(1+T)$$

$$\eta = 1 - In\lambda - 2\log E(1+T) - \log\beta - \log\alpha + \log\lambda + (\alpha-1)E(\log X) - (\alpha-1)\log(\lambda) + (\beta-1)E\log\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)$$

(37)

2.4. Maximum Likelihood Estimation of the EXDAL Distribution

$let\ x_1, x_2, x_3 \ldots . x_n$ be an independent random sample of size n from the EXDAL distribution with density function defined in equation (16), the likelihood function of the 4- parameter EXDAL distribution is given as

$$L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n}\left[\frac{\frac{\beta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{-\alpha-1}\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta-1}}{\left(1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right)^{2}}\theta exp\left[-\theta\left(\frac{\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)\right]\right] \quad (38)$$

The log–likelihood function is given as:

$$L = \sum_{i=1}^{n}\left\{\log\beta + \log\alpha - Log\lambda + (-\alpha-1)\log\left(\frac{x_i}{\lambda}\right) + (-\beta-1)\log\left[1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right] + \log\theta - \theta\left[\frac{\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{\beta}}{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{\beta}} - 2\log\left[1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right]\right]\right\} \quad (39)$$

Taking the partial derivative of the log-likelihood function w.r.t the parameters, we have:

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}Log\left[\frac{x_i}{\lambda}\right]+(-1-\beta)\sum_{i=1}^{n}-\log\left(\frac{x_i}{\lambda}\right)\left[\frac{x_i}{\lambda}\right]^{-\alpha}-\sum_{i=1}^{n}-\frac{2\beta\log\left(\frac{x_i}{\lambda}\right)\left[\frac{x_i}{\lambda}\right]^{-\alpha}\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-1-\beta}}{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}}-\theta\sum_{i=1}^{n}\frac{\beta\log\left(\frac{x_i}{\lambda}\right)\left[\frac{x_i}{\lambda}\right]^{-\alpha}\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-1-2\beta}}{\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right)^{2}}$$

$$+\frac{\beta\log\left(\frac{x_i}{\lambda}\right)\left[\frac{x_i}{\lambda}\right]^{-\alpha}\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-1-\beta}}{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}}$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)-\sum_{i=1}^{n}\frac{2\log\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)\right)^{-\beta}}{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}}-\theta\sum_{i=1}^{n}\left(-\frac{\log\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\left[1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right]^{-2\beta}\right)}{\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right)^{2}}-\frac{\log\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)\left[1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right]^{-\beta}}{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}}\right)$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\frac{\left(1+\left(\frac{\alpha}{\lambda}\right)^{-\alpha}\right)^{-\beta}}{1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}}$$

$$\frac{\delta L}{\delta \lambda}=-\frac{n}{\lambda}-\frac{n(-1-\alpha)}{\lambda}+(-1-\beta)\sum_{i=1}^{n}\frac{\alpha x_i\left(\frac{x_i}{\lambda}\right)^{-1-\alpha}}{\lambda^2}-\sum_{i=1}^{n}\frac{2\alpha\beta x_i\left(\frac{x_i}{\lambda}\right)^{-1-\alpha}\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-1-\beta}}{\lambda^2\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right)^{2}}-\theta\sum_{i=1}^{n}\left(-\frac{\alpha\beta x_i\left(\frac{x_i}{\lambda}\right)^{-1-\alpha}\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^{-1-2\beta}}{\lambda^2\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right)^{2}}-\frac{\alpha\beta x_i\left(\frac{x_i}{\lambda}\right)^{-1-\alpha}\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-1-\beta}}{\lambda^2\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{-\alpha}\right)^{-\beta}\right)}\right)$$

The maximum likelihood estimator $\hat{\varphi}$ of φ can be derived by solving the systems of non-linear equation ($\frac{\partial L}{\partial \alpha},\frac{\partial L}{\partial \beta},\frac{\partial L}{\partial \lambda} and \frac{\partial L}{\partial \theta})\frac{\partial \ell}{\partial \varphi}=0$. This equation can be solved using a numerical method known as Newton Raphson iterative scheme given by

$\hat{\varphi}=\varphi_q-H^{-1}(\varphi_q)U(\varphi_q)\hat{\varphi}=(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})^T$

where $U(\varphi_q)$ is the score function

$H^{-1}(\varphi_q)$ is the Hessian matrix, which is the second partial derivative of the log likelihood function defined by

$$H(\varphi_q)=\begin{bmatrix}\frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha\partial \beta} & \frac{\partial^2 l}{\partial \alpha\partial \lambda} & \frac{\partial^2 l}{\partial \alpha\partial \theta}\\ \frac{\partial^2 l}{\partial \beta\partial \alpha} & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta\partial \lambda} & \frac{\partial^2 l}{\partial \beta\partial \theta}\\ \frac{\partial^2 l}{\partial \lambda\partial \alpha} & \frac{\partial^2 l}{\partial \lambda\partial \beta} & \frac{\partial^2 l}{\partial \lambda^2} & \frac{\partial^2 l}{\partial \lambda\partial \theta}\\ \frac{\partial^2 l}{\partial \theta\partial \alpha} & \frac{\partial^2 l}{\partial \theta\partial \beta} & \frac{\partial^2 l}{\partial \theta\partial \lambda} & \frac{\partial^2 l}{\partial \theta^2}\end{bmatrix}$$

3. Results and Discussion

Table 1. Mean, Standard Deviation (SD) and the Median of the EXDAL Distribution

PARAMETER			α = 2			α = 4			α = 6		
θ	**λ**	**β**	**Mean**	**S.D**	**Median**	**Mean**	**S.D**	**Median**	**Mean**	**S.D**	**Median**
0.8	3	2	4.3787	8.9554	2.8375	1.0947	2.2388	0.7094	0.5473	1.1194	0.3547
	5	3	8.8155	13.4132	5.3164	2.2039	3.3533	1.3291	1.1019	1.6767	0.6645
	8	4	14.1576	19.4350	8.6975	3.7894	5.1087	2.1744	1.8947	2.5544	2.5544
2	3	2	4.4594	3.4368	4.5073	1.1148	0.8592	1.1268	0.5574	0.4296	0.5634
	5	3	8.0177	5.0854	7.8051	2.0044	1.2714	1.9513	1.0022	0.6357	0.9756
	8	4	13.1117	7.7563	12.5485	3.2779	1.9391	3.1371	1.6390	0.9695	1.5686
	3	2	5.1024	1.5839	5.3089	1.2756	0.3940	1.3272	0.6378	0.1980	0.6636
	5	3	8.7867	2.4166	9.0533	2.1967	0.6042	2.2633	1.0983	0.3021	1.1317
	8	4	14.1338	3.7683	14.5183	3.5334	0.9421	3.6296	1.7667	0.4710	1.8148

Table 2. Skewness and Kurtosis of the EXDAL Distribution

PARAMETER			α = 2		α = 4		α = 6	
θ	**λ**	**β**	**Sk**	**Ku**	**Sk**	**Ku**	**Sk**	**Ku**
0.8	3	2	0.9213	0.6715	0.9213	0.6715	0.7213	0.8715
	5	3	1.2279	1.1203	1.2279	1.1203	1.2279	1.1279
	8	4	1.4318	1.4410	1.4318	1.4410	1.4318	1.4410
2	3	2	-0.1320	-0.2269	-0.1320	-0.2269	-0.1320	-0.2269
	5	3	0.1603	-0.4626	0.1603	-0.4626	0.1603	-0.4626
	8	4	0.3169	-0.5454	0.3169	-0.5454	0.3169	-0.5454
5	3	2	-0.6830	0.3476	-0.6830	0.3476	-0.6830	0.3476
	5	3	-0.5324	0.0077	-0.5324	0.0077	-0.5324	0.0077
	8	4	-0.4620	-0.1501	-0.4620	-0.1501	-0.4620	-0.1501

Results from Table 1 clearly show that when the shape parameter α and the rate parameter θ are held constant, the mean, standard deviation, and median increases as the scale and shape parameter λ *and* β increases.
In Table 2, when the rate parameter θ, and the shape parameter α are held constant, an increase in the scale and shape parameter, increases the skewness and kurtosis.
We also illustrate the applicability of the EXDAL distribution using real life dataset. The dataset consists of the number of successive failures for the air conditioning system in a fleet of 13 Boeing 720 jet airplanes, [19]. The EXDAL distribution was compared with the Dagum and Log logistics distributions using both the Log-likelihood and Kolmogorov-Smirnov (K-S) tests.

Dataset

194	413	90	74	55	23	97	50	359	50	130	487
57	102	15	14	10	57	320	261	51	44	9	254
493	33	18	209	41	58	60	48	56	87	11	102
12	5	14	14	29	37	186	29	104	7	4	72
270	283	7	61	100	61	502	220	120	141	22	603
35	98	54	100	11	181	65	49	12	239	14	18
39	3	12	5	32	9	438	43	134	184	20	386
182	71	80	188	230	152	5	36	79	59	33	246
1	79	3	27	201	84	27	15	6	21	16	88
130	14	118	44	15	42	106	46	230	26	59	153
104	20	206	5	66	34	29	26	35	5	82	31
118	326	12	54	36	34	18	25	120	31	22	18
216	139	67	310	3	46	210	57	76	14	111	97
62	39	30	7	44	11	63	23	22	23	14	18
13	34	16	18	130	90	163	208	1	24	70	16
101	52	208	95	62	11	191	14	71			

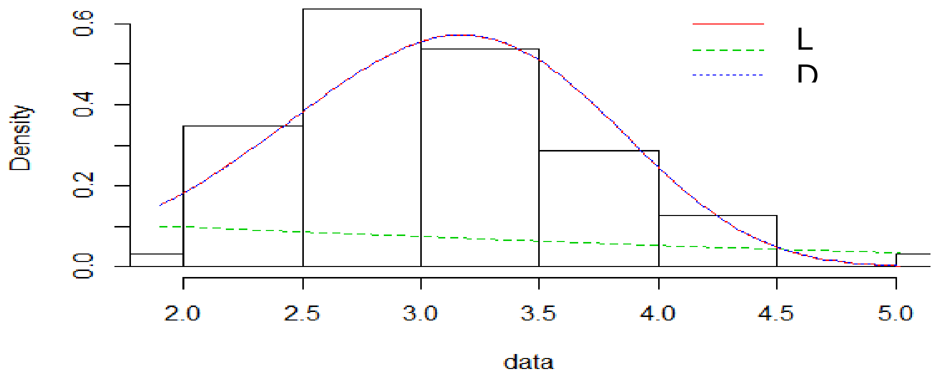
TABLE 3: Maximum Likelihood Estimates of the Log –Logistics, Dagum and EXDAL parameters

Distribution	**Log-logistics (LD)**	**Dagum (DD)**	**EXDAL**
Parameter Estimates	$\hat{\alpha}$ =0.6776 (0.1107) $\hat{\beta}$ =1.0411 (0.1873)	$\hat{\alpha}$ =0.5027 (0.4558) $\hat{\beta}$ =1.2901 (0.7494) $\hat{\lambda}$ =1.7243 (0.7981)	$\hat{\alpha}$= 0.8962 (0.0011) $\hat{\beta}$= 5.0126 (7.1227) $\hat{\lambda}$= 3.6182 (5.2654) $\hat{\theta}$ = 0.8351 (0.0082)
Log –likelihood	5.6714	5.6824	5.9027
K-S	0.1038	0.1034	0.0981

(Standard error of estimates in parenthesis)

Table 3 reveals that the Dagum, Log Logistics and EXDAL distributions can be used to model the dataset considered in this paper. However, the K-S and the log-likelihood tests show that the EXDAL distribution performs better than the Dagum and Log logistics distributions in fitting the dataset since it has the smallest K-S value and the largest log-likelihood value respectively.

Figure 5: Histogram plot and estimated densities for the LD, DD and EXDAL distributions for the dataset. Figure 5 clearly reveals the suitability of the EXDAL distribution for fitting the data considered.



4. Conclusion

This paper proposed the Exponential-Dagum Log logistics distribution. The mathematical properties of the proposed distribution have been carefully examined. The maximum likelihood estimation method was used in estimating the parameters of the proposed distribution. Finally, An application of the EXDAL distribution to a lifetime dataset shows its suitability in fitting lifetime dataset as it outperforms the Log-Logistics and the Dagum distribution in fitting the dataset considered.

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